AN

F. Cajori

ALGEBRA,

from Mr. Dixon

DESIGNED FOR THE USE OF

HIGH SCHOOLS, ACADEMIES, AND COLLEGES.

BY

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PREFACE.

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THE present volume is designed as the first part of a complete work upon the science of Algebra. Several chapters written for this volume have been omitted for want of room. These chapters embrace *Indeterminate Analysis*, *Permutations*, *Combinations*, *Calculus of Probabilities*, *Continued Fractions*, and about fifty pages on the *General Theory of Equations*, including *Sturm's Theorem*, *Horner's Method of Resolving Numerical Equations of all Degrees*, and *Cardan's Formula for Cubic Equations*, which will be inserted in the next volume. The present volume, however, will be found to contain as much as the great majority of students in our High Schools and Academies generally study, and the student who thoroughly masters this volume will have acquired sufficient algebraic knowledge to enable him to pursue the remainder of the usual mathematical course.

The arrangement of this work is, in many respects, new. The plan of treating every subject as the solution of a general problem, or as the demonstration of a theorem, it is hoped will greatly facilitate rigid class

instruction. Many of the demonstrations have been rendered so clear that they may be readily comprehended by students of ordinary capacity; others which are of a more abstruse character will require close application. The student should early endeavor to acquire the ability to grasp algebraic principles in all their generality, and hence no principle has been given with a mere *il*lustration, which is too often considered a *dem*onstration.

The attention of teachers is called to the *classification of Algebraic Symbols* in Chap. I.; the *explanation of Subtraction* and Articles (81), (82), (83), (84), (94), (95), (96), (97), in Chap. II.; Articles (112), (113), and (114) in Chapter III.; the *demonstration of the rule for finding the Greatest Common Divisor of two polynomials* in Chap. IV.; Articles (160) and (162) in Chap. VI.; the *general Discussion of the Courier Problem* in Chap. X.; and the *Demonstration of the Multinomial Theorem* in Chap. XVIII. The method of solving equations of the third and fourth degree as set forth in Articles (335), (337), (354), and (355), although *tentative* in its character and not *practically* general, furnishes the means of resolving many problems which have heretofore been considered difficult. This method is considered valuable in an educational rather than in a scientific point of view.

An unusual number of practical examples have been inserted in this volume on the principle that algebraic

skill can only be acquired by extensive practice. Other peculiarities might be mentioned, but it is deemed unnecessary.

The work is now submitted to an intelligent public with the hope that after a careful examination it will meet with the approbation of all those who favor the use of text books that do not attempt to simplify by the omission of what is difficult.

GREENMOUNT, NEAR RICHMOND, INDIANA,
Dec., 1856.

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ALGEBRA.

CHAPTER I.

DEFINITIONS.

(1.) ALGEBRA is a general method of investigating the relations of quantities.

(2.) *Quantity* is that which admits of increase, or of diminution.

(3.) *Algebraic notation* is the method of representing by symbols algebraic quantities, their relations, and the operations to be performed upon them.

(4.) *Algebraic symbols* are characters used in algebraic expressions.

(5.) *Algebraic symbols* are of six kinds; namely,

Symbols of *Quantity*,
" " *Operation*,
" " *Relation*,
" " *Aggregation*,
" " *Continuation*,
and " " *Deduction*.

SYMBOLS OF QUANTITY.

(6.) The *symbols of quantity*, generally used, are the Arabic numerals and alphabetic characters.

(7.) The Arabic numerals, 1, 2, 3, 4, 5, &c., are used to represent known quantities.

(8.) The first letters of the alphabet, a, b, c, &c., are generally used to represent known quantities, or quantities that may be assumed to be of any value whatever.

(9.) The final letters of the alphabet, x, y, z, u, &c., are generally used to represent quantities which depend for their values upon

known quantities, or upon quantities which may be assumed to be known.

(10.) The symbol 0 is called *zero*, and denotes the absence of quantity, or that which is less than any assignable quantity.

(11.) The symbol ∞ is called *infinity*, and denotes that which is greater than any assignable quantity.

(12.) The notations, a', a'', a''', a'''', &c., and $a_1, a_2, a_3, a_4, \ldots\ldots a_n$, are often used to denote different quantities which occupy similar positions in different operations. a' is read *a prime;* a'', *a second;* a''', *a third;* a'''', *a fourth;* &c., and a_1, is read *a sub one;* a_2, *a sub two;* a_3, *a sub three;* a_4, *a sub four* . . . . a_n, *a sub n.*

(13.) The symbols $'$, $''$, $'''$, $''''$, &c., are called *accents.* The symbols $_1, _2, _3, _4, \ldots\ldots _n$, are called *subscripts.*

SYMBOLS OF OPERATION.

(14.) The *symbols of operation*, are $+$, $-$, $\sim$, $\times$, $\cdot$, $\div$, $:$, $-$, $)$, $\underline{|\ \ }$, ^{2}, ^{3}, ^{4}, &c., $^{\frac{1}{2}}$, $^{\frac{1}{3}}$, $^{\frac{1}{4}}$, &c., $\sqrt{\ }$, $\sqrt[3]{\ }$, $\sqrt[4]{\ }$, &c.

(15.) The symbol $+$ is called *plus*, and is the sign of addition. Thus, $a+b$ indicates the addition of a and b.

(16.) The symbol $-$ is called *minus*, and is the sign of subtraction. Thus, $a-b$ indicates the subtraction of b from a.

(17.) The symbol $\sim$, when placed between two quantities, denotes that the less is to be subtracted from the greater. Thus, $6\sim5$, or $5\sim6$, denotes that 5 is to be subtracted from 6; and $a\sim b$ denotes that b is to be subtracted from a, or a from b, according as a is greater or less than b.

(18.) The symbols $\times$ and $\cdot$ are signs of multiplication. Thus, 7×5, or $7\cdot5$, indicates that 7 is to be multiplied by 5, or 5 by 7; $a\times b$, or $a\cdot b$, indicates that a is to be multiplied by b, or b by a.

(19.) In representing the multiplication of literal quantities, the sign of multiplication is generally omitted. Thus, instead of $a\times b$, or $a\cdot b$, we write ab; and instead of $2\times a$, or $2\cdot a$, we write $2a$.

(20.) The symbols $\div$, $:$, $-$, $)$, and $\underline{|\ \ }$, are signs of division. Thus, $6\div3$, $6:3$, $\frac{6}{3}$, $3)6$, or $6\underline{|3}$, indicates the division of 6 by 3; also, $a\div b$, $a:b$, $\frac{a}{b}$, $b)a$, or $a\underline{|b}$, indicates the division of a by b.

(21.) The symbols ^{1}, ^{2}, ^{3}, ^{4}, &c., are called *exponents*, and are signs of involution. Thus, a^5, which is an abbreviation of $aaaaa$, denotes that a is to be raised to the fifth power, that is, a is to be taken five times as a factor. The exponent ^{1} is not usually written.

(22.) The symbols $^{\frac{1}{2}}$, $^{\frac{1}{3}}$, $^{\frac{1}{4}}$, &c., are called *fractional exponents*, and are the signs of evolution. Thus, $a^{\frac{1}{2}}$, $a^{\frac{1}{3}}$, $a^{\frac{1}{4}}$, &c., respectively indicate that the square root, the cube root, the fourth root, &c., of a are to be extracted.

(23.) In fractional exponents the numerator denotes a *power*, and the denominator a *root*. Thus, $x^{\frac{3}{5}}$ denotes the fifth root of the third power of x, or the third power of the fifth root of x.

(24.) The symbol $\sqrt{}$ is called the *radical*, and is the sign of evolution. Thus, $\sqrt[2]{a}$, $\sqrt[3]{a}$, $\sqrt[4]{a}$, &c., respectively indicate that the square root, the cube root, the fourth root, &c., of a are to be extracted. The small figure in the angle of the radical is the index of the root. When no index is written, ^{2} is understood. Thus, $\sqrt{a}$ is the same as $\sqrt[2]{a}$.

SYMBOLS OF RELATION.

(25.) The *symbols of relation* are $:$, $=$, $::$, $>$ $<$, $\div$ . . ., and $\div\!\!\div$: : : .

(26.) The symbol : denotes ratio. Thus, $a : b$ denotes the ratio of a to b.

(27.) The symbols $=$ and $::$ are signs of equality. Thus, $a=b$ denotes that a equals b; and $a:b=c:d$, or $a:b::c:d$ denotes that the ratio of a to b equals the ratio of c to d.

The symbol $::$ is not used except to denote the equality of ratios. Thus, we never write $a::b$ for $a=b$.

(28.) The symbols $>$ and $<$ are signs of inequality. Thus, $a>b$ denotes that a is greater than b; and $a<b$, that a is less than b.

(29.) The symbol $\div$ . . . is the sign of an *arithmetical series*. Thus, $\div\, a.b.c.d$ denotes the equality of the difference between a and b, b and c, and c and d.

(30.) The symbol $\div\!\!\div$: : : is the sign of a *geometrical series*. Thus, $\div\!\!\div\, a:b:c:d$ denotes the equality of the ratios of a to b, b to c, and c to d.

SYMBOLS OF AGGREGATION.

(31.) The *symbols of aggregation* are $\overline{\quad}$, |, ( ), [ ], and { }.

(32.) The symbol $\overline{\quad}$ is called a *vinculum*, and denotes that the quantities over which it is placed are to be considered as one quantity. Thus, $\sqrt{a+b+c}$ denotes that the square root of the sum of a, b, and c is to be extracted.

(33.) The symbol | is called a *bar*, and denotes that the quantities, in the column immediately preceding it, are to be considered as one quantity. Thus, $\left.\begin{matrix}+a\\+b\\+c\end{matrix}\right| \begin{matrix}x\\ \\ \\ \end{matrix}$ denotes that the sum of a, b, and c is to be multiplied by x.

(34.) The *parenthesis* ( ), *brackets* [ ], and *braces* { }, denote that the quantities contained within them are to be considered as one quantity. Thus, $(b+c)x$ denotes that the sum of b and c is to be multiplied by x; $[a+(b+c)x]y$ denotes that the sum of a and $(b+c)x$ is to be multiplied by y; and $\{z+[a+(b+c)x]y\}u$ denotes that the sum of z and $[a+(b+c)x]y$ is to be multiplied by u.

SYMBOLS OF CONTINUATION.

(35.) The *symbols of continuation* are . . . . and - - - -, and are equivalent to *&c.*, *and so on*, or *continued according to the same law.* Thus, a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, &c., and a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, &c., may be written a, a^2, a^3, ... and a_1, a_2, a_3, ...

SYMBOLS OF DEDUCTION.

(36.) The *symbols of deduction* are $\therefore$ and $\because$

(37.) The symbol $\therefore$ signifies *therefore*, *whence*, *consequently*, *hence*, *from which we infer*, &c.

(38.) The symbol $\because$ signifies *since*, or *because.*

(39.) A *monomial*, or *term*, is an algebraic expression which is not connected to any other by the sign of addition or subtraction; as, a, $2a$, ab, a^2bc, $\frac{a}{b}$, $(a+b)$, $(x+y)x$, $(x+a)$ $(y+b)$, &c.

(40.) A *binomial* is an algebraic expression which is composed of two terms; as, $a+b$, $2x-3y$, $(x+y)+z$, $(x+a)+(y+b)$, &c.

(41.) A *trinomial* is an algebraic expression which is composed of three terms; as, $a+b+c$, $x-y+3z$, $(x+y)+a+(c+d)$, $(x-y)-(a-b)+(c-d)$, &c.

(42.) A *polynomial*, or *multinomial*, is an algebraic expression which is composed of several terms; as $a+b$, $a-b+c$, $x+y-z+n$, $a+b-c+d-e$, &c.

(43.) A *residual* is an algebraic expression which denotes the difference of two terms; as, $a-b$, $(a+b)-(c+d)$, &c.

(44.) A *simple term* is one which contains but one sign of addition or subtraction expressed or understood; as, $-3x$, $4ab$, &c. But $(a+b)$ is not a simple term, because it is equivalent to $(+a+b)$ which contains two *plus* signs.

(45.) A *compound term* is one which is composed of two or more simple terms affected by a sign of aggregation; as, $(a-b)$, $\sqrt{2x-3y+4z}$, $\{(x+y)z+(a+b)[c+(m-n)]+q\}$, &c.

(46.) A *positive term* is one that is affected by the sign of addition; as, $+6x$, $+(a+b)$, $+(x-y)$, or $6x$, $(a+b)$, $(x-y)$, &c.

(47.) A *negative term* is one that is affected by the sign of subtraction; as, $-6a$, $-(x-y)$, $-(3x-4y+7z)$, &c.

(48.) *Like*, or *similar terms*, are those that contain the same letters affected by the same exponents; numerals denoting abstract numbers; and numerals denoting concrete numbers of the same denomination. Thus $6a^2x^3$ and $59a^2x^3$, and $4xy^{\frac{3}{2}}$ and $-7xy^{\frac{3}{2}}$, $-6(a^2+b^3)$ and $42(a^2+b^3)$, 67 and 34, \$12 and $-$\$10 are respectively similar.

(49.) *Unlike*, or *dissimilar terms*, are those that are not similar. Thus, a and x, a^2x and ax^2, $-6(x+y)$ and $-7(x+z)$, are respectively dissimilar.

(50.) *Homogeneous terms* are those in which the sum of the exponents of the literal factors in each are equal. Thus, $2x^2y$ and xy^2, $4a^3b^2$ and $5ab^4$, are respectively homogeneous.

(51.) The *coefficient* of a term is that factor which is considered as denoting the number of times the rest of the term is taken. Thus, in $6a$, 6 is the coefficient of a; in ax, a may be considered the coefficient of x, or 1 may be considered the coefficient of ax; also, in $6ax$, $6a$ is the coefficient of x, and 6 the coefficient of ax.

(52.) A *numerical quantity* is one which is expressed by numerals; as, 6, 43, &c.

(53.) A *literal quantity* is one which is expressed entirely or in part by letters; as, ax, $3ab$, &c.

(54.) A *rational quantity* is one which can be expressed without the aid of a radical sign or fractional exponent; as, a, $\sqrt{a^2}=a$, $b^{\frac{4}{2}}=b^2$, $\sqrt{36}=6$, &c.

(55.) An *irrational quantity* is one which can not be expressed without the aid of a radical sign or fractional exponent; as, $\sqrt{a}$, $b^{\frac{1}{2}}$, $\sqrt{2}$, &c.

(56.) The *power* of a quantity is the product resulting by taking the quantity a certain number of times as a factor. Thus, $aaaa$ or a^4 denotes the 4th power of a.

(57.) The *root* of a quantity is a quantity which taken a certain number of times as a factor, produces the given quantity.

(58.) The *reciprocal* of a quantity is unity divided by that quantity. Thus, $\frac{1}{a}$ is the reciprocal of a, and $\frac{1}{\frac{a}{b}}$, or $\frac{b}{a}$ is the reciprocal of $\frac{a}{b}$.

(59.) An *algebraic formula* is the general expression of a mathematical truth.

(60.) A *proposition* is something proposed to be done, or to be demonstrated.

(61.) A *problem* is something proposed to be done.

(62.) A *theorem* is something proposed to be demonstrated.

(63.) A *lemma* is something to be demonstrated in order to render what follows more easy.

(64.) A *corollary* or *consectary* is an obvious consequence deduced from some preceding truth or demonstration.

(65.) A *scholium* is a remark appended to the demonstration of a theorem or to the solution of a problem.

(66.) An *hypothesis* is a supposition made in the enunciation of a proposition, or in the course of a demonstration.

(67.) A *direct demonstration* is one in which the reasoning employed is regular deduction.

(68.) An *indirect demonstration* is one by which a thing is proved to be true by showing that the supposition that it is not true leads to an absurdity. This kind of demonstration is frequently called a *reductio ad absurdum*.

(69.) An *axiom* is a self-evident truth.

AXIOMS.

1. The whole is equal to the sum of all its parts.

2. If equal quantities be added to equal quantities, the sums will be equal.

3. If equal quantities be subtracted from equal quantities, the remainders will be equal.

4. If equal quantities be multiplied by equal quantities, the products will be equal.

5. If equal quantities be divided by equal quantities, the quotients will be equal.

6. If unequal quantities be added to equal quantities, the sums will be unequal.

7. If unequal quantities be subtracted from equal quantities, the remainders will be unequal.

8. If unequal quantities be multiplied by equal quantities, the products will be unequal.

9. If unequal quantities be divided by equal quantities, the quotients will be unequal.

10. Quantities which are an equal number of times the same quantity, are equal to each other.

11. Equal powers of equal quantities are equal.

12. Equal roots of equal quantities are equal.

(70.) EXERCISES IN NOTATION.

1. Write a added to b.

2. Write a subtracted from b.

3. Write the difference between a and b. *Ans.* $a \sim b$.

4. Write in three ways a multiplied by b.

5. Write in two ways 6 multiplied by 8.

6. Write in five ways a divided by b.

7. Write in three ways the product of a added to b by the difference between b and a when a is greater than b.

8. Write in two ways the square root of a, added to b.

9. Write in two ways the cube root of a, added to the square of the sum of a, b, and c.

10. Write in two ways the square root of the sum of a and b.

11. Write in two ways the cube root of the sum of a and the square of the sum of a, b, and c.

12. Write the product of the sum of a and b by the sum of x and y, divided by the cube root of a, diminished by d.

(71.) EXERCISES IN NUMERATION.

1. Read $a+b$.

2. Read $a-b$.

3. Read $a \sim c$.

4. Read 6×7 or $6 \cdot 7$.

5. Read $a \times b$, or $a \cdot b$.

6. Read ab.

7. Read $a \div b$.

8. Read $a:b$. *Ans.* a divided by b, or the ratio of a to b.

9. Read a^2. *Ans.* a squared, a square, or a's square.

10. Read $\sqrt{a}$.

11. Read $\sqrt{a+b}$.

12. Read $\sqrt{a+\sqrt{b+c}}$.

13. Read $6\sqrt{a+\sqrt{b+\sqrt[3]{c}}}$.

14. Read $3\sqrt[4]{a^{\frac{1}{3}}+\sqrt[3]{a^{\frac{1}{2}}+\sqrt{a}}}$.

15. Read $(x^5-y^7)\left\{a+b[a+3(b-p)]+4y^2\right\}$.

CHAPTER II.

ADDITION.

(72.) *Addition* is finding the simplest expression for the sum of several algebraic quantities.

CASE I.

(73.) When quantities are entirely dissimilar.

RULE.

Connect them together by their proper signs.

PROBLEM.

Find the sum of a, $-b$, $+3c$, and $-5d$.

SOLUTION.

Connecting these expressions together, we have, $a-b+3c-5d$.

EXAMPLES.

1. Find the sum of $6a$ and $-7b$. *Ans.* $6a-7b$.

2. Find the sum of $4a$, $-3b$, and $+5x$. *Ans.* $4a-3b+5x$.

3. Find the sum of ax, $+bm$, $-cy$, and $+nw$.

Ans. $ax+bm-cy+nw$.

CASE II.

(74.) When the quantities are similar and have the same sign.

RULE

1. *When the quantities are numerical, add as in arithmetic, and prefix the sign* $+$, *or* $-$, *as the case may be.*

RULE

2. *When the quantities are literal, add the coefficients, affix the literal part, and prefix* $+$, *or* $-$, *as the case may be.*

PROBLEM

1. Find the sum of -4, -7, -8, and -3.

SOLUTION.

Operation.

$$\begin{array}{r} -4 \\ -7 \\ -8 \\ -3 \\ \hline -22 \end{array}$$

Adding these numerals together and prefixing the sign —, we have -22.

PROBLEM

2. Find the sum of $-3abx$, $-7abx$, $-6abx$, and $-9abx$.

SOLUTION.

Operation.

$$\begin{array}{r} -3abx \\ -7abx \\ -6abx \\ -9abx \\ \hline -25abx \end{array}$$

Adding the numerical coefficients, prefixing the sign —, and annexing the common literal factor, we have $-25abx$.

EXAMPLES.

1.	2.	3.	4.	5.
$4a$	$4ab$	$-abc$	$5(a+b)$	$2\sqrt{a+x^2}$
a	$7ab$	$-3abc$	$7(a+b)$	$3\sqrt{a+x^2}$
$6a$	$5ab$	$-4abc$	$3(a+b)$	$5\sqrt{a+x^2}$
a	$2ab$	$-12abc$	$14(a+b)$	$7\sqrt{a+x^2}$
$2a$	ab	$-2abc$	$(a+b)$	$9\sqrt{a+x^2}$
$8a$	$3ab$	$-14abc$	$12(a+b)$	$10\sqrt{a+x^2}$
$3a$	$11ab$			
*$25a$	$33ab$	$-36abc$	$42(a+b)$	

6.	7.	8.
$-4(a^2-b^2)^{\frac{1}{3}}$	$3(x^2+y)^{\frac{1}{2}}$	$4(x^2-y^2)^{\frac{1}{3}}$
$-7(a^2-b^2)^{\frac{1}{3}}$	$4\sqrt{x^2+y}$	$6\sqrt[3]{x^2-y^2}$
$-11(a^2-b^2)^{\frac{1}{3}}$	$5\sqrt{x^2+y}$	$9\sqrt[3]{x^2-y^2}$
$-8(a^2-b^2)^{\frac{1}{3}}$	$14(x^2+y)^{\frac{1}{2}}$	$\sqrt[3]{x^2-y^2}$
$-3(a^2-b^2)^{\frac{1}{3}}$	$16(x^2+y)^{\frac{1}{2}}$	$8(x^2-y^2)^{\frac{1}{3}}$

* When no sign is written + is understood.

9.	10.	11.
$3a^2+\ \ p$	$2x^2y-\ y^2$	$(a^2-b^2)^{\frac{1}{2}}-16\sqrt{m^2-n^2}$
$7a^2+\ 3p$	$3x^2y-4y^2$	$2\sqrt{a^2-b^2}\ -\ 4\sqrt{m^2-n^2})$
$9a^2+\ 6p$	$5x^2y-6y^2$	$14\sqrt{a^2-b^2}\ -\ 7(m^2-n^2)^{\frac{1}{2}}$
$a^2+\ 9p$	$6x^2y-3y^2$	$3(a^2-b^2)^{\frac{1}{2}}-\ 5(m^2-n^2)^{\frac{1}{2}}$
$14a^2+15p$	$8x^2y-4y^2$	$5(a^2-b^2)^{\frac{1}{2}}-\ 8(m^2-n^2)^{\frac{1}{2}}$

12.

$$
\begin{array}{l}
3a^2-\ \ 7+5abc\\
4a^2-\ \ 6+7abc+4x^2y\\
5a^2-\ \ 3+8abc+5x^2y+4mn\\
6a^2-\ \ 4+6abc+6x^2y+\ mn+\ cd\\
7a^2-12+3abc+4x^2y+3mn+4cd\\
\hline
\end{array}
$$

CASE III.

(75.) When the quantities are similar, and all have not the same sign.

RULE.

Find the sum of the similar positive terms, also the sum of the similar negative terms, and then, disregarding the signs, ascertain their difference, and prefix +, *or* —, *according as the sum of the positive or negative terms is greater.*

PROBLEM.

Find the sum of $6a$, $-7a$, $-3a$, $+4a$, $+2a$.

SOLUTION.

Operation.

$$
\begin{array}{r}
+6a\\
-7a\\
-3a\\
+4a\\
-2a\\
\hline
-2a
\end{array}
$$

The sum of the positive terms is $+10a$, and the sum of the negative terms is $-12a$. Disregarding the signs, we have $2a$ for the difference between $12a$ and $10a$, to which prefix —, because $12a$ is greater than $10a$ and has a *minus* sign.

We may also add the terms successively. Thus, $-2a+4a$ is $+2a$; which, added to $-3a$, is $-a$; which, added to $-7a$, is $-8a$; which, added to $+6a$, is $-2a$.

EXAMPLES.

1. What is the sum of $-4xy$, $+7xy$, $+6xy$, and $-5xy$? *Ans.* $4xy$.

2. What is the sum of $-6ab$, $-7ab$, $+3ab$, and $+4ab$? *Ans.* $-6ab$.

3. What is the sum of $3a+xy$, $-4a-4xy$, $+7a-5xy$, and $6a+xy$? *Ans.* $12a-7xy$.

4. What is the sum of $4xy-ab$, $-3xy+4ab$, $-4xy-5ab$, and $+5xy+4ab$? *Ans.* $2xy+2ab$.

5. What is the sum of $a+b+c+d+e-f$, $a+b+c+d-e+f$, $a+b+c-d+e+f$, $a+b-c+d+e+f$, $a-b+c+e+f$, and $-a+b+c+e+f$? *Ans.* $4a+4b+4c+2d+4e+4f$.

6. What is the sum of $7a-5c+3b$ and $2a-3c-7b$? *Ans.* $9a-8c-4b$.

7. What is the sum of $5a+4b-3c-7d+8$ and $3a-12b+7c-10d-4$? *Ans.* $8a-8b+4c-17d+4$.

8. What is the sum of $-7f+3a$, $4f-2a$, $3f-3a$, and $+2a$? *Ans.* 0.

9. What is the sum of $12h-3c-7f+3g$ and $-3h+8c-2f-9g+5x$? *Ans.* $9h+5c-9f-6g+5x$.

10. What is the sum of $16a-5b+15c-9d$, $3a+18b-5c-7d+3e$, $-7a-2b-3d+5e-9h$, and $11a-3b+2c+8d+7h$? *Ans.* $23a+8b+12c-11d+8e-2h$.

11. What is the sum of $8a+b$, $2a-b+c$, $-3a+5b+2d$, $-6b-3c+3d$, and $-5a+7c-2d$? *Ans.* $2a-b+5c+3d$.

12. What is the sum of $-a+3b-c-115d+6e-5f$, $3a-2b-3c-d+27e$, $5b-8c+3e-7f$, $7a-6b+17c+9d-5e+11f$, and $-3a-5c-2d+6e-9f+g$? *Ans.* $6a-109d+37e-10f+g$.

13. What is the sum of $\sqrt{x^2+y^2}-m^2+n^2-2mn$, $-\sqrt{x^2+y^2}+3m^2-3n^2+5mn$, $-5\sqrt{x^2+y^2}-4m^2+5n^2-7mn$, $2(x^2+y^2)^{\frac{1}{2}}+12m^2-2\frac{1}{2}n^2+mn$, and $8(x^2+y^2)^{\frac{1}{2}}-8m^2-\frac{1}{2}n^2-6mn$? *Ans.* $5\sqrt{x^2+y^2}+2m^2-9mn$.

14. What is the sum of $5ax^{\frac{1}{2}}-\sqrt[3]{x+y}+(a-b)$, $-7a\sqrt{x}+2(x+y)^{\frac{1}{3}}$

$-3(a-b)$, $12a\sqrt{x}-3\sqrt[3]{x+y}+12(a-b)$, $-3a\sqrt{x}-4\sqrt[3]{x+y}-(a-b)$, and $-ax^{\frac{1}{2}}+(x+y)^{\frac{1}{3}}-3(a-b)$?

Ans. $6ax^{\frac{1}{2}}-5\sqrt[3]{x+y}+6(a-b)$.

15. What is the sum of $2\sqrt{xy+xz+yz}+\sqrt[4]{ax+by}$, $5\sqrt{xy+xz+yz}-3(ax+by)^{\frac{1}{4}}$, $12(xy+xz+yz)^{\frac{1}{2}}+5(ax+by)^{\frac{1}{4}}$, $-3\sqrt{xy+xz+yz}-2\sqrt[4]{ax+by}$, and $(xy+xz+yz)^{\frac{1}{2}}+(ax+by)^{\frac{1}{4}}$?

Ans. $17\sqrt{ax+xz+yz}+2\sqrt[4]{ax+by}$.

16. What is the sum of $4(a+b)\sqrt{x^2-y^2}-2(a-b)\sqrt{x^2+y^2}$, $-3(a+b)\sqrt{x^2-y^2}+(a-b)\sqrt{x^2+y^2}$, $-(a+b)(x^2-y^2)^{\frac{1}{2}}+3(a-b)(x^2+y^2)^{\frac{1}{2}}$, $6(a+b)(x^2-y^2)^{\frac{1}{2}}-(a-b)(x^2+y^2)^{\frac{1}{2}}$, $10(a+b)\sqrt{x^2-y^2}-5(a-b)(x^2+y^2)^{\frac{1}{2}}$, and $-2(a+b)(x^2-y^2)^{\frac{1}{2}}+4(a-b)\sqrt{x^2+y^2}$?

Ans. $14(a+b)\sqrt{x^2-y^2}$.

17. What is the sum of $10\sqrt{2}+5\sqrt[7]{8}-7\sqrt[m]{5}+2\sqrt[3]{a}$, $5\sqrt{2}+\sqrt[7]{8}+4\sqrt[m]{5}-3\sqrt[3]{a}$, and $-3\sqrt{2}-9\sqrt[7]{8}-3\sqrt[m]{5}+\sqrt[3]{a}+\sqrt{ab}$?

Ans. $12\sqrt{2}-3\sqrt[7]{8}-6\sqrt[m]{5}+\sqrt{ab}$.

18. What is the sum of $5a^4b+3a^{-2}b^2c-7ab$, $-6a^4b+2a^{-2}b^2c+17ab$, and $9a^4b-8a^{-2}b^2c-10ab$? *Ans.* $8a^4b-3a^{-2}b^2c$.

19. What is the sum of $-3(ax+by+cz)^{\frac{1}{4}}-\sqrt{x^2+y^2}+a-b$, $2\sqrt[4]{ax+by+cz}+(x^2+y^2)^{\frac{1}{2}}-3(a-b)$, $\sqrt[4]{ax+by+cz}-\sqrt{x^2+y^2}+2(a-b)$, $3\sqrt[4]{ax+by+cz}+(x^2+y^2)^{\frac{1}{2}}+a-b$, $5\sqrt[4]{ax+by+cz}+(x^2+y^2)^{\frac{1}{2}}-2(a-b)$, and $(ax+by+cz)^{\frac{1}{4}}-\sqrt{x^2+y^2}-3(a-b)$?

Ans. $9(ax+by+cz)^{\frac{1}{4}}-4(a-b)$.

20. What is the sum of $6a^{\frac{1}{3}}b^{\frac{2}{3}}-9c^{\frac{2}{3}}d+10a^{\frac{1}{2}}b^{\frac{3}{4}}$, $-6a^{\frac{1}{2}}b^{\frac{3}{4}}-a^{\frac{1}{3}}b^{\frac{2}{3}}+6c^{\frac{2}{3}}d$, $2c^{\frac{2}{3}}d-3a^{\frac{1}{2}}b^{\frac{3}{4}}-3a^{\frac{1}{3}}b^{\frac{2}{3}}$, and $-2a^{\frac{1}{3}}b^{\frac{2}{3}}+c^{\frac{2}{3}}d-a^{\frac{1}{2}}b^{\frac{3}{4}}$? *Ans.* 0.

CASE IV.

(76.) When the quantities are partially similar, and have the same or different signs.

RULE.

Find the expression for the sum of the coefficients of the terms, and annex a part, or all of what is common to each term, according as the addition is to be partial or complete.

REMARK.—We may take as many factors of any term for the coefficient as we choose.

PROBLEM.

1. Find an expression for the partial addition of $6max$, $3nax$, $12pax$, and $9vax$.

SOLUTION.

Considering x as the common factor, the coefficients are $6ma$, $3na$, $12pa$, and $9va$, whose sum may be represented by $(6ma+3na+12pa+9va)$, to which annexing x, we have $(6max+3na+12pa+9va)x$ for an expression of the partial addition of $6max+3nax+12pax+9vax$.

Considering ax as the common factor, the coefficients are $6m$, $3n$, $12p$, and $9v$, whose sum may be represented by $(6m+3n+12p+9v)$, to which annexing ax, we have $(6m+3n+12p+9v)ax$ for another expression of the partial addition of $6max+3nax+12pax+9vax$.

PROBLEM

2. Find an expression for the complete addition of $6max$, $3nax$ $12pax$, and $9vax$.

SOLUTION.

We observe that $6max = 2m \cdot 3ax$,
$3nax = n \cdot 3ax$,
$12pax = 4p \cdot 3ax$,
and $9vax = 3v \cdot 3ax$.

From this we see that $3ax$ is the entire common factor to which prefixing the expression for the sum of the coefficients $2m$, n, $4p$, and $3v$, gives $(2m+n+4p+3v)3ax$, or $3(2m+n+4p+3v)ax$ for the expression of the complete additon of $6max$, $3nax$, $12pax$, and $9vax$.

EXAMPLES.

1. Find the sum of ax, bx, and cx. *Ans.* $(a+b+c)x$.

2. Find the sum of $ax+by+cz$, $bx+cy+az$, and $cx+ay+bz$. *Ans.* $(a+b+c)(x+y+z)$.

3. Find the sum of $6x^2y+7x^2z+9x^2ym$. *Ans.* $[3(2+3m)y+7z]x^2$.

4. Find the sum of $(a-b)\sqrt{x}+(m-n)\sqrt{y}+\sqrt{2}$, $(a+c)x^{\frac{1}{2}}-(m-n)y^{\frac{1}{2}}+2\sqrt{2}$, $(b-c)\sqrt{x}+3(m-n)\sqrt{y}-3\sqrt{2}$, and $(c-a)\sqrt{x}-5(m-n)\sqrt{y}-6\sqrt{2}$. *Ans.* $(a+c)\sqrt{x}-2(m-n)\sqrt{y}-6\sqrt{2}$.

5. Find the sum of $(m+n)y^2-(a-b)x^2+axy$, $(n-p)y^2-(2a+b)x^2-bxy$, $(p-2n)y^2-(c-3a)x^2+cxy$, and $(q-m)y^2-(c+2d)x^2-dxy$.
Ans. $qy^2-2(c+d)x^2+(a-b+c-d)xy$.

6. Find the sum of ax^2+by+c and dx^2+hy+k.
Ans. $(a+d)x^2+(b+h)y+c+k$.

7. Find the sum of x^2+xy+y^2, $ax^2-axy+ay^2$, and $-by^2+bxy+bx^2$.
Ans. $(1+a+b)x^2+(1-a+b)xy+(1+a-b)y^2$.

8. Find the sum of $(a+b)x+(c-d)y-x\sqrt{2}$, $(a-b)x+(3c+2d)y+5x\sqrt{2}$, $2bx+3dy-2x\sqrt{2}$, and $-3bx-dy-4x\sqrt{2}$.
Ans. $(2a-b)x+(4c+3d)y-2x\sqrt{2}$.

9. Find the sum of $5a^mb^p+3a^{-3}b^{m-1}-3a^3$, $-3ca^mb^p+4g^2a^{-3}b^{m-1}-a+10a^3$, and $a^mb^p+a+3a^2b^2-2g^2a^{-3}b^{m-1}$.
Ans. $(6-3c)a^mb^p+(2g^2+3)a^{-3}b^{m-1}+7a^3+3a^2b^2$.

10. Find the sum of $3\cdot2^{-7}+5^6$, $-8\cdot2^{-7}+3a^nb^{-m}$, and $-13\cdot5^6+4a\cdot2^{-7}+ca^nb^{-m}$.
Ans. $(4a-5)2^{-7}-12.5^6+(c+3)a^nb^{-m}$.

11. Find the sum of $9a^{-3}b^{-2}c^4-7b$, $18b-a^nb^m+c^x-3\cdot2^5$, and $3a^nb^m-ha^{-3}b^{-2}c^4+3c^x-5\cdot2^5$.
Ans. $(9-h)a^{-3}b^{-2}c^4+2a^nb^m+11b+4c^x-8\cdot2^5$.

12. Find the sum of $(a+b)\sqrt{x}+(2+m)\sqrt{y}$, $4y^{\frac{1}{2}}+(a+c)x^{\frac{1}{2}}$, $3n\sqrt{y}+(2d-e)x^{\frac{1}{2}}$, $(m+n)y^{\frac{1}{2}}+(b+2c)\sqrt{x}$, and $-2n\sqrt{x}+12a\sqrt{y}$.
Ans. $[2(a+b+d-n)+3c-e]\sqrt{x}+2[3+m+2(n+3a)]\sqrt{y}$.

SUBTRACTION.

(77.) Subtraction is finding the simplest expression for the difference between two algebraic quantities.

CASE I.

(78.) When the terms are entirely dissimilar.

RULE.

Write the quantity to be subtracted after the one from which it is to be subtracted with the sign — between them.

PROBLEM

1. Subtract $+b$ from a; also, $-b$ from a.

SOLUTION.

By the rule we have for the subtraction of $+b$ from a, $a-(+b)$ which is the same as $a-b$. That $a-(+b)$ is the same as $a-b$, may be proved in the following manner:

Operation.

$$\begin{array}{l} a+b-b \\ \underline{+b} \\ a-b \end{array}$$

Since $a+b-b$ is equal to a; if from $a+b-b$, $+b$ be taken, the remainder will be the same as when $+b$ is taken from a. But, $+b$ taken from $a+b-b$ leaves $a-b$; therefore, $+b$ taken from a leaves $a-b$; that is, $a-(+b)=a-b$.

By the rule we have for the subtraction of $-b$ from a, $a-(-b)$ which is the same as $a+b$. That $a-(-b)$ is the same as $a+b$, may be proved in the following manner:

Operation.

$$\begin{array}{r} a+b-b \\ \underline{-b} \\ a+b \end{array}$$

Since, $a+b-b$ is equal to a; if from $a+b-b$, $-b$ be taken, the remainder will be the same as when $-b$ is taken from a. But $-b$ taken from $a+b-b$ leaves $a+b$; therefore, $-b$ taken from a leaves $a+b$; that is, $a-(-b)=a+b$.

PROBLEM

2. Subtract $b-c$ from a.

SOLUTION.

Operation.

$$\begin{array}{lcr} a+b-b+c-c \\ +b -c \\ \hline a -b+c \end{array}$$

Since, $a+b-b+c-c$ is equal to a; if from $a+b-b+c-c$, $b-c$ be taken, the remainder will be the same as when $b-c$ is taken from a. But, $b-c$ taken from $a+b-b+c-c$ leaves $a-b+c$; therefore, $b-c$ taken from a leaves $a-b+c$.

EXAMPLES.

1. Subtract $c+d$ from $a+b$. *Ans.* $a+b-c-d$.
2. Subtract $b-c-d+e$ from a. *Ans.* $a-b+c+d-e$.
3. Subtract $-(b+c+d)$ from a. *Ans.* $a+b+c+d$.
4. Subtract $-y-z$ from x. *Ans.* $x+y+z$.

CASE II.

(79.) When the terms are similar, or partially similar.

RULE.

Imagine the sign of the term to be subtracted to be MINUS *when it is* $+$, *and* PLUS *when it is* $-$, *and then proceed as in addition.*

REMARK.—The reason of this rule has been made manifest in the solution of problems 1 and 2, Case I.

PROBLEM.

Subtract $6ax-9yz$ from $7ax+4yz$.

SOLUTION.

Operation.

$$\begin{array}{r} 7ax+\ \ 4yz \\ 6ax-\ \ 9yz \\ \hline ax+13yz \end{array}$$

Imagining $6ax-9yz$ to be $-6ax+9yz$ and adding it to $7ax+4yz$, we have $ax+13yz$, the remainder.

EXAMPLES.

1.

$$\begin{array}{ll} \text{From} & 4a+3b-2c+8d \\ \text{Take} & \underline{\ \ a+2b+\ \ c+5d} \\ \text{Rem.} & 3a+b-3c+3d \end{array}$$

2.

$$\begin{array}{ll} \text{From} & 12xy+3y^2-17x^2--3\sqrt{2} \\ \text{Take} & \underline{-5xy+7y^2-19x^2+2\sqrt{2}} \\ \text{Rem.} & 17xy-4y^2+\ \ 2x^2-5\sqrt{2} \end{array}$$

3.

$$\begin{array}{ll} \text{From} & 28ax^3-16a^2x^2+25a^3x-13a^4 \\ \text{Take} & \underline{18ax^3+20a^2x^2-24a^3x-\ \ 7a^4} \\ \text{Rem.} & 10ax^3-36a^2x^2+49a^3x-\ \ 6a^4 \end{array}$$

4.

$$\begin{array}{ll} \text{From} & 2(a+b)+3(a-x) \\ \text{Take} & \underline{\ \ (a+b)-3(a-x)} \\ \text{Rem.} & \ \ a+b\ +6(a-x). \end{array}$$

5. From $\sqrt{x^2-y^2}+4(x+y)-3\sqrt{a+x}$ subtract $3(x+y)-2(x^2-y^2)^{\frac{1}{2}}+3(a+x)^{\frac{1}{2}}$. *Ans.* $3\sqrt{x^2-y^2}+(x+y)-6\sqrt{a+x}$.

6. From $x^2-2xy+(x^2+y^2)+(2xy-y^2)$ subtract $x^2+2xy-y^2+(x^2+y^2)-2(2xy-y^2)$. *Ans.* $y^2-4xy+3(2xy-y^2.)$

7. From $2a^2+ax+x^2-12a^2x+20ax^2-4x^3+6a^2x^2-10ax^3$ subtract $a^2-3ax+2x^2-16a^2x+12ax^2-12ax^3-4x^3+2a^2x^2$.

Ans. $a^2+4ax-x^2+4a^2x+8ax^2+2ax^3+4a^2x^2$.

8. From $4y^2-4yx+x^2-2a(x+y)+6\sqrt{a^2-x^2}-8\sqrt{b^2-y^2}$ take $4x^2-4xy+y^2-4a(x+y)-10\sqrt{b^2-y^2}+4\sqrt{a^2-x^2}$.

Ans. $3y^2-3x^2+2a(x+y)+2\sqrt{a^2-x^2}+2\sqrt{b^2-y^2}$.

9. From $(a+b)\sqrt{x^2+y^2}+(a+c)(a+x)^3$ take $(a-b)\sqrt{x^2+y^2}+c(a+x)^3$. *Ans.* $2b\sqrt{x^2+y^2}+a(a+x)^3$.

10. From $ax^2+byx+cy^2$ take $dx^2-hxy+ky^2$.

Ans. $(a-d)x^2+(b+h)xy+(c-k)y^2$.

11. From $a(x+y)-bxy+c(x-y)$ take $4(x+y)+(a+b)xy-7(x-y)$.

Ans. $(a-4)(x+y)-(a+2b)xy+(c+7)(x-y)$.

12. From $2x-y+(y-2x)-(x-2y)$ take $y-2x-(2y-x)+(x+2y)$.

Ans. $y-x$.

13. From $\sqrt{x^2-y^2}-2(a+x)^{\frac{1}{2}}+3$ take $-3\sqrt{a+x}+4(x^2-y^2)^{\frac{1}{2}}-1$.

Ans. $\sqrt{a+x}-3\sqrt{x^2-y^2}+4$.

14. From $2x(x+y)^{\frac{1}{4}}-3axy+2abc$ take $-17axy+11abc-x\sqrt[4]{x+y}$.

Ans. $14axy-9abc+3x(x+y)^{\frac{1}{4}}$.

15. From $a(x-y)^{\frac{1}{2}}+bxy+c(a+x)^2$ take $(x-y)^{\frac{1}{2}}-bxy+(a+c)(a+x)^2$.

Ans. $(a-1)(x-y)^{\frac{1}{2}}+2bxy-a(a+x)^2$.

16. From $(a+b)(x+y)-(c+d)(x-y)+m$ take $(a-b)(x+y)+(c-d)(x-y)-n$.

Ans. $2b(x+y)-2c(x-y)+m+n$.

17. From $ax^2+mxy+nx+b$ take $sx^2-pxy+qx-c$.

Ans. $(a-s)x^2+(m+p)xy+(n-q)x+b+c$.

18. From $(a-b)xy-(p+q)\sqrt{x+y}-hx^2$ take $(2p-3q)(x+y)^{\frac{1}{2}}-axy-(3+h)x^2$.

Ans. $(2a-b)xy-(3p-2q)\sqrt{x+y}+3x^2$.

19. From $9a^mx^2-13+20ab^3x-4b^mcx^2$ take $3b^mcx^2+9a^mx^2-6+3ab^3x$.

Ans. $17ab^3x-7b^mcx^2-7$.

20. From $5a^4-7a^3b^2-3c^{-1}d^2+7d$ take $-15a^3b^2+3a^4-3a^2-7c^{-1}d^2$.

Ans. $2a^4+8a^3b^2+4c^{-1}d^2+7d+3a^2$.

MULTIPLICATION.

(80.) MULTIPLICATION is finding an expression for the product of two or more algebraic quantities.

PROPOSITION

(81.) 1. *When a positive quantity is multiplied by a positive quantity, the product is positive.*

DEMONSTRATION.

Operation.

$$\left.\begin{array}{r}+(+2)\\+(+2)\\\underline{+(+2)}\end{array}\right\}\text{ or }\left.\begin{array}{r}+2\\+2\\\underline{+2}\end{array}\right\}\text{ or }\begin{array}{r}+2\\\underline{+3}\end{array}$$

$$+(+6) \;=\; +6 \;=\; +6$$

Let us multiply $+2$ by $+3$. This means that the $+2$ is to be taken positively, or additively, 3 times. Hence, the result is $+6$, as is shown by the operation.

The principle contained in this proposition is generally expressed in the following

RULE.

Plus $(+)$ *multiplied by plus* $(+)$ *gives plus* $(+)$.

PROPOSITION

(82.) 2. *When a negative quantity is multiplied by a positive quantity, the product is negative.*

DEMONSTRATION.

Operation.

$$\left.\begin{array}{r}+(-2)\\+(-2)\\\underline{+(-2)}\end{array}\right\}\text{ or }\left.\begin{array}{r}-2\\-2\\\underline{-2}\end{array}\right\}\text{ or }\begin{array}{r}-2\\\underline{+3}\end{array}$$

$$+(-6) \;=\; -6 \;=\; -6$$

Let us multiply -2 by $+3$. This means that the -2 is to be taken positively, or additively, 3 times. Hence, the result is -6, as is shown by the operation.

The principle contained in this proposition is generally expressed in the following

RULE.

Minus $(-)$ *multiplied by plus* $(+)$ *gives minus* $(-)$.

PROPOSITION

(83.) 3. *When a positive quantity is multiplied by a negative quantity, the product is negative.*

DEMONSTRATION.

Operation.

$$\left.\begin{array}{r}-(+2)\\-(+2)\\\underline{-(+2)}\end{array}\right\}\text{ or }\left.\begin{array}{r}-2\\-2\\\underline{-2}\end{array}\right\}\text{ or }\begin{array}{r}+2\\\underline{-3}\end{array}$$

$$-(+6) \;=\; -6 \;=\; -6$$

Let us multiply $+2$ by -3. This means that the $+2$ is to be taken negatively, or subtractively, 3 times. Hence, the result is -6, as is shown by the operation.

Operation.

$$\begin{array}{r} +2-2 \\ +2 \\ \hline -2 \end{array}$$

The expression $-(+2)$ is the same as -2, because $-(+2)$ means that $+2$ is to be subtracted. But there is nothing from which to subtract it. Let us, then, subtract it from nothing, or zero. For zero, we write $+2-2$, and taking $+2$ from it we have -2, as is shown by the operation. Therefore, $-(+2)=-2$.

The principle contained in this proposition is generally expressed in the following

RULE.

Plus $(+)$ *multiplied by minus* $(-)$ *gives minus* $(-)$.

PROPOSITION

(84.) 4. *When a negative quantity is multiplied by a negative quantity, the product is positive.*

DEMONSTRATION.

Operation.

$$\left.\begin{array}{r} -(-2) \\ -(-2) \\ -(-2) \end{array}\right\} \text{ or } \left.\begin{array}{r} +2 \\ +2 \\ +2 \end{array}\right\} \text{ or } \begin{array}{r} -2 \\ -3 \\ \hline +6 \end{array}$$

$$\begin{array}{lll} -(-6) & +6 & \end{array}$$

Let us multiply -2 by -3. This means that the -2 is to be taken negatively, or subtractively, 3 times. Hence, the result is $+6$, as is shown by the operation.

Operation.

$$\begin{array}{r} +2-2 \\ -2 \\ \hline +2 \end{array}$$

The expression $-(-2)$ is the same as $+2$, because $-(-2)$ means that -2 is to be subtracted. But there is nothing from which to subtract it. Let us, then, subtract it from nothing, or zero. For zero, we write $+2-2$, and taking -2 from it we have $+2$, as is shown by the operation. Therefore, $-(-2)=+2$.

The principle contained in this proposition is generally expressed in the following

RULE.

Minus $(-)$ *multiplied by minus* $(-)$ *gives plus* $(+)$.

(85.) The principles contained in these four propositions may be expressed by the following

RULE.

The multiplication of LIKE *signs gives* PLUS $(+)$, *and the multiplication of* UNLIKE *signs,* MINUS $(-)$.

PROPOSITION

(86.) 5. *The product of two literal terms may be expressed by writing them in order, with or without a sign of multiplication between the consecutive terms, preceded by* $+$ *or* $-$, *according as the signs are like or unlike.*

DEMONSTRATION.

The truth of this proposition depends upon **(19.)** Thus, a multiplied by $-b$ may be expressed by $-a\times b$, $-a\cdot b$, or $-ab$; $(a+b)$ multiplied by $(c-d)$ may be expressed by $(a+b)\times(c-d)$, $(a+b)\cdot(c-d)$, or $(a+b)(c-d)$; and $(a+3)$ multiplied by $(4-b)$ may be expressed by $(a+3)\times(4-b)$, $(a+3)\cdot(4-b)$, or $(a+3)(4-b)$.

PROPOSITION

(87.) 6. *If two terms, when the exponent and sign of each are not considered, have a common part, their product may be expressed by the common part affected by the sum of their exponents, and preceded by* $+$, *or* $-$, *according as the signs of the terms are like or unlike.*

DEMONSTRATION.

The two terms $+a^2$ and $-a^3$, when the exponents and signs are not considered, have a common part a; therefore, we are to prove that the product of $+a^2$ and $-a^3$ is $-a^5$. Since, $+a^2=+aa$ and $-a^3=-aaa$, we know by the last proposition the product of $+aa$ and $-aaa$ is $-aaaaa$. But, $-aaaaa=-a^5$. Therefore, $+a^2\times -a^3=-a^{2+3}=-a^5$.

REMARK.—By this proposition, we have $+2^1\times 2^1=+2^{1+1}=+2^2=4$; $-a^{-3}\times -a^{-1}=+a^{-3-1}=+a^{-4}$; $a^{-2}\times -a^3=-a^{-2+3}=-a^1$ or $-a$.

GENERAL RULE.

(88.) *In multiplication, coefficients are multiplied, and exponents are added.*

CASE I.

(89.) When both multiplicand and multiplier are monomials.

RULE.

Multiply according to the principles of the preceding propositions

EXAMPLES.

1. Multiply $4a^2b^2cd$ by $3abc^2d^2$. *Ans.* $12a^3b^3c^3d^3$.

2. Multiply $12\sqrt{ay}$ by $4bx$. *Ans.* $48bx\sqrt{ay}$.

3. Multiply $5\frac{1}{2}x^2y^3z^4$ by $6xy^4z^3$. *Ans.* $33x^3y^7z^7$.

4. Multiply $13a^2b^3x^3y$ by $-5abxy^3$. *Ans.* $-65a^3b^4x^4y^4$.

5. Multiply $-20a^pb^q$ by $5a^mb^nc^r$. *Ans.* $-100a^{m+p}b^{n+q}c^r$.

6. Multiply a^m by a^n. *Ans.* a^{m+n}.

7. Multiply a^m by a^{-n}. *Ans.* a^{m-n}.

8. Multiply a^{-m} by a^n. *Ans.* a^{n-m}.

9. Multiply a^{-m} by a^{-n}. *Ans.* $a^{-(m+n)}$.

10. Multiply $2a^{-3}$, $7a^{-9}$, and $-3a^6$ together. *Ans.* $-42a^{-6}$.

11. Multiply $3\cdot7^{-9}$, 7^{-2}, and $4\cdot7^8$ together. *Ans.* $12\cdot7^{-3}$.

12. Multiply $-7a^{-1}b^4c^{-5}$ by $3a^2b^{-5}c$. *Ans.* $-21ab^{-1}c^{-4}$.

13. Multiply $-a^{p-q}$, $-3a^{3q-2}f$, and $5a^{p+7}cx$ together.
Ans. $15a^{2p+2q+5}fcx$.

14. Multiply $-13a^{-1}c^{-3}$ by $-4a^{-3}b^{-6}c^2$. *Ans.* $52a^{-4}b^{-6}c^{-1}$.

15. Multiply $a^{-m}b^p$, a^nb^{-r}, and $a^{n+m}b$ together. *Ans.* $a^{2n}b^{p-r+1}$.

16. Multiply $(a+y)^{-3}h^5l^4$, $(a+y)^{m+3}l^{-4}m$, and $(a+y)$ together.
Ans. $mh^5(a+y)^{m+1}$.

17. Multiply $a^{\frac{2}{3}}$ by $a^{\frac{1}{3}}$. *Ans.* a.

18. Multiply $a^{\frac{3}{4}}$ by $-a^{-\frac{1}{2}}$. *Ans.* $-\sqrt[4]{a}$.

19. Multiply $a^{\frac{1}{2}}$ by $a^{\frac{1}{3}}$. *Ans.* $a^{\frac{5}{6}}$.

20 Multiply $-x^{-\frac{5}{8}}$ by $x^{\frac{3}{7}}$. *Ans.* $-a^{-\frac{11}{56}}$.

CASE II.

(90.) When the multiplicand is a polynomial, and the multiplier a monomial.

RULE.

Multiply each term of the multiplicand by the multiplier, connecting them by their proper signs.

PROBLEM.

Multiply $6a+4b^2c-3d^3$ by $4a^2$.

SOLUTION.

Operation.

$$\begin{array}{l} 6a + \ 4b^2c \ \ -3d^3 \\ 4a^2 \\ \hline 24a^3+16a^2b^2c-12a^2b^3 \end{array}$$

Multiplying $6a$, $+4b^2c$, and $-3d^3$ by $4a^2$, respectively, gives $24a^3$, $+16a^2b^2c$, and $-12a^2d^3$ which connected by their proper signs is $24a^3+16a^2b^2c-12a^2d^3$.

EXAMPLES.

1. Multiply $a^2-3ab-5b^2$ by $4a^2b$. *Ans.* $4a^4b-12a^3b^2-20a^2b^3$.

2. Multiply $2a^3b^5-5a^2c^6+9a^3b^2c^3$ by $3a^2bc^2$.
Ans. $6a^5b^6c^2-15a^4bc^8+27a^5b^3c^5$.

3. Multiply $2a^2-3c+5$ by bc. *Ans.* $2a^2bc-3bc^2+5bc$.

4. Multiply ax^3-bx^2+cx-d by $-x^5$.
Ans. $-ax^8+bx^7-cx^6+dx^5$.

5. Multiply $5mn+3m^2-2n^2$ by $12abn$.
Ans. $60abmn^2+36abm^2n-24abn^3$.

6. Multiply $3ax-5by+7xy$ by $-7abxy$.
Ans. $-21a^2bx^2y+35ab^2xy^2-49abx^2y^2$.

7. Multiply $-15a^2b+3ab^2-12b^3$ by $-5ab$.
Ans. $75a^3b^2-15a^2b^3+60ab^4$.

8. Multiply $a^mx^n+b^my^n-c^ny^m-d^nx^m$ by x^my^n.
Ans. $a^mx^{m+n}y^n+b^mx^my^{2n}-c^nx^my^{m+n}-d^nx^{2m}y^n$.

9. Multiply $3x^{-2}-5x^my^{-n}+z^{-1}$ by $2x^{-4}y^m$.
Ans. $6x^{-6}y^m-10x^{m-4}y^{m-n}+2x^{-4}y^mz^{-1}$.

10. Multiply $2a^{-\frac{2}{3}}-7x^{-\frac{1}{2}}y^4-11c^{\frac{1}{6}}$ by $axy^{-3}c^{\frac{5}{6}}$.
Ans. $2a^{\frac{1}{3}}xy^{-3}c^{\frac{5}{6}}-7ax^{\frac{1}{2}}yc^{\frac{5}{6}}-11axy^{-3}c$.

CASE III.

(91.) When both the multiplicand and multiplier are polynomials

RULE.

Multiply each term of the multiplicand by each term of the multiplier, and add the products.

PROBLEM.

Multiply a^2+ab+b^2 by $a+b$.

SOLUTION.

Operation.

$$\begin{array}{l} a^2+\ \ ab+b^2 \\ \underline{a+b\qquad\qquad} \\ a^3+\ \ a^2b+\ \ ab^2 \\ \underline{\quad +\ \ a^2b+\ \ ab^2+b^3} \\ a^3+2a^2b+2ab^2+b^3 \end{array}$$

Multiplying a^2+ab+b^2 by a gives $a^3+a^2b+ab^2$; also, multiplying it by b gives $a^2b+ab^2+b^3$ which results added produces $a^3+2a^2b+2ab^2+b^3$.

EXAMPLES.

1. Multiply $a+b$ by $a+b$. *Ans.* $a^2+2ab+b^2$.
2. Multiply $a-b$ by $a-b$. *Ans.* $a^2-2ab+b^2$.
3. Multiply $a+b$ by $a-b$. *Ans.* a^2-b^2.
4. Multiply a^2-ab+b^2 by $a+b$. *Ans.* a^3+b^3.
5. Multiply a^2+ab+b^2 by $a-b$. *Ans.* a^3-b^3.
6. Multiply $a^3-a^2b+ab^2-b^3$ by $a+b$. *Ans.* a^4-b^4.
7. Multiply $a^3+a^2b+ab^2+b^3$ by $a-b$. *Ans.* a^4-b^4.
8. Multiply a^4-2b^3 by $a-b$. *Ans.* $a^5-2ab^3-a^4b+2b^4$.
9. Multiply x^2-3x-7 by $x-2$. *Ans.* x^3-5x^2-x+14.
10. Multiply $a^2+a^4+a^6$ by a^2-1. *Ans.* a^8-a^2.
11. Multiply $4a^2-16ax+3x^2$ by $5a^3-2a^2x$.
Ans. $20a^5-88a^4x+47a^3x^2-6a^2x^3$.
12. Multiply $a^4-2a^3b+4a^2b^2-8ab^3+16b^4$ by $a+2b$.
Ans. a^5+32b^5.
13. Multiply $\frac{5}{2}x^2+3ax-\frac{7}{3}a^2$ by $2x^2-ax-\frac{1}{2}a^2$.
Ans. $5x^4+\frac{7}{2}ax^3-\frac{107}{12}a^2x^2+\frac{5}{6}a^3x+\frac{7}{6}a^4$.
14. Multiply $15a^{-6}b^2-7a^{-5}b^4+6a^{-4}b^6$ by $8a^{-2}b^2-3a^{-1}b^4$.
Ans. $120a^{-8}b^4-101a^{-7}b^6+69a^{-6}b^8-18a^{-5}b^{10}$.
15. Multiply $a^m+b^p-2c^n$ by $2a^m-3b$.
Ans. $2a^{2m}+2a^mb^p-4a^mc^n-3a^mb-3b^{p+1}+6bc^n$.
16. Multiply $x^{-3p}+3a^mx^{-2p}-10a^{2m}x^{-p}$ by $a^2x^q+5a^{m+2}x^{q+p}-2a^{2m+2}x^{q+2p}$.
Ans. $a^2x^{q-3p}+8a^{m+2}x^{q-2p}+3a^{2m+2}x^{q-p}-56a^{3m+2}x^q+20a^{4m+2}x^{q+p}$.
17. Multiply $3x+6$, $3x+2$, $3x-2$, and $3x-6$ together.
Ans. $81x^4-360x^2+144$.

18. Multiply $3a-2b$, $4a-3b$, $4a+3b$, and $3a+2b$ together.

Ans. $144a^4-145a^2b^2+36b^4$.

19. Multiply $x-a$, $x-b$, and $x-c$ together.

Ans. $x^3-(a+b+c)x^2+(ab+ac+bc)x-abc$.

20. Multiply $3a^{\frac{2}{3}}b^{-\frac{1}{2}}+7a^{-\frac{1}{4}}b^{\frac{3}{5}}$ by $4a^{\frac{1}{5}}b^{-\frac{1}{3}}c^4-3a^3b^{-\frac{7}{8}}d^{-3}$.

Ans. $4(3a^{\frac{13}{15}}b^{-\frac{5}{6}}+7a^{-\frac{1}{20}}b^{\frac{4}{15}})c^4-3(3a^{\frac{11}{3}}b^{-\frac{11}{8}}+7a^{\frac{11}{4}}b^{-\frac{11}{40}})d^{-3}$.

MULTIPLICATION BY DETACHED COEFFICIENTS.

PROBLEM.

(92.) To multiply by detached coefficients.

RULE.

Arrange the multiplicand and multiplier according to the ascending or descending powers of a particular letter, and then remove the letters and multiply the coefficients thus detached and restore the letters according to the law of exponents in each particular case.

DEMONSTRATION.

Let us multiply $x^3+x^2y+xy^2+y^3$ by $x-y$. The terms in these polynominals as they stand are arranged according to the descending powers of x and the ascending powers of y. Removing the letters, we have the coefficients $1+1+1+1$ to be multiplied by $1-1$. The product, as shown in the operation, is $1+0+0+0-1$. We know that the exponent of x in the first term of the product of the given polynominal must be ^{4}. Annexing the letters to the coefficients $1+0+0+0-1$ according to the descending powers of x and the ascending powers of y, we have, $1x^4+0x^3y+0x^2y^2+0xy^3-1y^4$, or simply x^4-y^4.

Operation.

$$\begin{array}{l}
1+1+1+1\\
\underline{1-1}\\
1+1+1+1\\
\underline{-1-1-1-1}\\
1+0+0+0-1
\end{array}$$

Again, let us multiply $2a^3-3ab^2+5b^3$ by $2a^2-5b^2$. Arranging

Operation.

$$\begin{array}{l} 2+0-\ \ 3+\ \ 5 \\ 2+0-\ \ 5 \\ \hline 4+0-\ \ 6+10 \\ \qquad -10-\ \ 0+15-25 \\ \hline 4+0-16+10+15-25 \end{array}$$

the terms according to the descending powers of a or the ascending powers of b, we have, $2a^2+0a^2b-3ab^2+5b^3$ and $2a^2+0ab-5b^2$. The product of $2+0-3+5$ by $2+0-5$ is $4+0-16+10-15-25$, to which annexing the letters we have, $4a^5+0a^4b-16a^3b^2+10a^2b^3+15ab^4-25b^5$, or $4a^5-16a^3b^2+10a^2b^3+15ab^4-25b^5$.

EXAMPLES.

1. Multiply $3a^2+4ax-5x^2$ by $2a^2-6ax+4x^2$.
Ans. $6a^4-10a^3x-22a^2x^2+46ax^3-20x^4$.

2. Multiply x^3-3x^2+3x-1 by x^2-2x+1.
Ans. $x^5-5x^4+10x^3-10x^2+5x-1$.

3. Multiply $y^2-ya+\frac{1}{4}a^2$ by $y^2+ya-\frac{1}{4}a^2$.
Ans. $y^4-a^2y^2+\frac{1}{2}a^3y-\frac{1}{16}a^4$.

4. Multiply $x^4-x^3+x^2-x+1$ by cx^3-bx^2+ax.
Ans. $cx^7-(b+c)x^6+(a+b+c)x^5-(a+b+c)x^4+(a+b+c)x^3-(a+b)x^2+ax$.

5. Multiply $a^8-a^7b+a^6b^2-a^5b^3+a^4b^4-a^3b^5+a^2b^6-ab^7+b^8$ by $a-b$.
Ans. a^9-b^9.

6. Multiply $x^4+4x^3y+6x^2y^2+4xy^3+y^4$ by $x^3+3x^2y+3xy^2+y^3$.
Ans. $x^7+7x^6y+21x^5y^2+35x^4y^3+35x^3y^4+21x^2y^5+7xy^6+y^7$.

DIVISION.

(93.) DIVISION is finding a factor of a given quantity which multiplied into a given factor will produce the given quantity, or is finding how many times one quantity is contained in another.

PROPOSITION

(94.) 1. *When a positive quantity is divided by a positive quantity, the quotient is positive.*

DEMONSTRATION.

Let us divide $+6$ by $+2$. Here we seek a factor which multiplied into $+2$ will produce $+6$. The sign of the factor sought must be $+$, or like the sign of the 2, in order that this factor multiplied into $+2$ shall produce a positive quantity. Thus $+6 \div +2$ can not be equal to -3, because $+2 \times -3 = -6$; but, $+6 \div +2 = +3$, because $+2 \times +3 = +6$.

Operation.

$$\frac{+6}{+2} = \frac{\cancel{+2} \times +3}{\cancel{+2}} = +3$$

Again; the division of $+6$ by $+2$ can be represented thus, $\frac{+6}{+2}$. Factoring the numerator, we have, $\frac{+2 \times +3}{+2}$; and canceling the $+2$ in both terms, we obtain, $+3$.

The principle contained in this proposition is generally expressed by the following

RULE.

Plus $(+)$ *divided by plus* $(+)$ *gives plus* $(+)$.

PROPOSITION

(95.) 2. *When a negative quantity is divided by a positive quantity, the quotient is negative.*

DEMONSTRATION.

Let us divide -6 by $+2$. Here we seek a factor which multiplied into $+2$ will produce -6. The sign of the factor sought must be $-$, or unlike the sign of the 2, in order that this factor multiplied into $+2$ shall produce a negative quantity. Thus, $-6 \div +2$ can not be equal to $+3$, because $+2 \times +3 = +6$; but, $-6 \div +2 = -3$, because $-2 \times -3 = -6$.

Operation.

$$\frac{-6}{+2} = \frac{\cancel{+2} \times -3}{\cancel{+2}} = -3.$$

Again; the division of -6 by $+2$ can be represented thus, $\frac{-6}{+2}$. Factoring the numerator, we have $\frac{+2 \times -3}{+2}$; and canceling the $+2$ in both terms, we obtain, -3.

The principle contained in this proposition is generally expressed by the following

RULE.

Minus $(-)$ *divided by plus* $(+)$ *gives* $(-)$.

PROPOSITION

(96.) 3. *When a positive quantity is divided by a negative quantity, the quotient is negative.*

DEMONSTRATION.

Let us divide $+6$ by -2. Here we seek a factor which multiplied into -2 will produce $+6$. The sign of the factor sought must be $-$, or like the sign of the 2, in order that this factor multiplied into -2 shall produce a positive quantity. Thus, $+6\div-2$ can not be equal to $+3$, because $-2\times+3=-6$; but, $+6\div-2=-3$, because $-2\times-3=+6$.

Operation.

$$\frac{+6}{-2}=\frac{\cancel{-2}\times-3}{\cancel{-2}}=-3.$$

Again; the division of $+6$ by -2 can be represented thus, $\frac{+6}{-2}$. Factoring the numerator, we have, $\frac{-2\times-3}{-2}$; and canceling the -2 in both terms, we obtain -3.

The principle contained in this proposition is generally expressed by the following

RULE.

Plus $(+)$ *divided by minus* $(-)$ *gives minus* $(-)$.

PROPOSITION

(97.) 4. *When a negative quantity is divided by a negative quantity, the quotient is positive.*

DEMONSTRATION.

Let us divide -6 by -2. Here we seek a factor which multiplied into -2 will produce -6. The sign of the factor sought must be $+$, or unlike the sign of the 2, in order that this factor multiplied into -2 shall produce a negative quantity. Thus, $-6\div-2$ can not be equal -3, because $-2\times-3=+6$; but, $-6\div-2=+3$, because $-2\times+3=-6$.

Operation.

$$\frac{-6}{-2} = \frac{-\cancel{2} \times +3}{-\cancel{2}} = +3.$$

Again; the division of $-6 \div -2$ can be represented thus, $\frac{-6}{-2}$. Factoring the numerator we have $\frac{-2 \times +3}{-2}$; and canceling the -2 in both terms, we obtain $+3$.

The principle contained in this proposition is generally expressed by the following

RULE.

Minus $(-)$ *divided by minus* $(-)$ *gives plus* $(+)$.

(98.) The principles contained in these four propositions are expressed by the following

RULE.

The division of like signs gives plus $(+)$, *and of unlike signs* $(-)$.

REMARK.—We may consider that a positive quotient denotes the number of times that the divisor must be *subtracted* from the dividend to obtain zero for a remainder. Thus, we see that $+2$ must be *subtracted* three times from $+6$ and -2 three times from -6, to obtain zero. Hence the quotients obtained by dividing $+6$ by $+2$, and -6 by -2 ought both to have the same sign; and, therefore, the quotient in each case must be $+3$.

Operation.

$$\begin{array}{rr} +6 & -6 \\ \underline{+2} & \underline{-2} \\ +4 & -4 \\ \underline{+2} & \underline{-2} \\ +2 & -2 \\ \underline{+2} & \underline{-2} \\ 0 & 0 \end{array}$$

We may also consider that a negative quotient denotes the number of times the divisor must be *added* to the dividend to obtain zero for a remainder. Thus, we see that $+2$ must be *added* three times to -6, and -2, *three* times to $+6$, to obtain zero. Hence, the quotients obtained by dividing -6 by $+2$, and $+6$ by -2, ought both to have the same sign; and therefore, the quotient in each case must be -3.

Operation.

$$\begin{array}{rr} -6 & +6 \\ \underline{+2} & \underline{-2} \\ -4 & +4 \\ \underline{+2} & \underline{-2} \\ -2 & +2 \\ \underline{+2} & \underline{-2} \\ 0 & 0 \end{array}$$

PROPOSITION

(99.) 5. *The quotient obtained by dividing one term by another, may be expressed by those factors of the dividend which are not common to the divisor.*

DEMONSTRATION.

Let us divide $6abcd$ by $2bc$. The dividend $6abcd = 2bc \times 3ad$, from which we see that 3, a, and d are the factors of the dividend which are not common to the divisor. Therefore, $3ad$ is the quotient. Also the quotient of $aaaaa$ divided by aaa is aa, because aaa of the dividend is the same as the divisor, thus leaving aa which is not common.

PROPOSITION

(100.) 6. *If two terms, when the exponent and the sign of each are not considered, have a common part, the quotient arising from dividing one by the other may be expressed by the common part affected by an exponent equal to the exponent of the dividend minus the exponent of the divisor, and preceded by + or —, according as the signs of the dividend and divisor are like or unlike.*

DEMONSTRATION.

The two terms, $-a^5$ and $+a^3$, when the exponents and signs are not considered, have a common part a. We are now to prove that the quotient arising from dividing $-a^5$ by $+a^3$ is $-a^2$. Since $-a^5 = -aaaaa$, and $+a^3 = aaa$, we know by the last proposition that the quotient arising from dividing $-aaaaa$ by $+aaa$ is $-aa$. But $-aa = -a^2$; therefore the proposition is proved.

By this proposition, we have $\frac{a^8}{a^3} = a^{8-3} = a^5$; $\frac{a^m}{a^n} = a^{m-n}$; $\frac{a^3}{a^{-2}} = a^{3-(-2)}$ $= a^{3+2} = a^5$; $\frac{a^3}{a^4} = a^{3-4} = a^{-1}$

PROPOSITION

(101.) 7. *Any factor may be transferred from the denominator to the numerator of a fraction, or from the numerator to the denominator, by changing the sign of the exponent.*

DEMONSTRATION.

We have just seen $\frac{a^8}{a^3}=a^{8-3}$. But $a^{8-3}=\frac{a^8a^{-3}}{1}$; therefore, $\frac{a^8}{a^3}=\frac{a^8a^{-3}}{1}$; or, in other words, a^8 has been transferred from the denominator of $\frac{a^8}{a^3}$ to the numerator by changing the sign of the exponent.

Again; dividing both numerator and denominator of $\frac{a^8}{a^3}$ by a^8, and we have $\frac{1}{a^3\div a^8}=\frac{1}{a^{3-8}}$. But $a^{3-8}=a^3a^{-8}$; therefore, $\frac{a^8}{a^3}=\frac{1}{a^3a^{-8}}$, or, in other words, a^8 has been transferred from the numerator to the denominator by changing the sign of the exponent.

By this proposition, we have $\frac{a}{b}=\frac{ab^{-1}}{1}=ab^{-1}$; $\frac{a}{b}=\frac{1}{a^{-1}b}$; $\frac{a}{b}=\frac{b^{-1}}{a^{-1}}$; $\frac{a^3b^2}{cd^4}=a^3b^2c^{-1}d^{-4}$; $\frac{1}{a}=a^{-1}$; $\frac{1}{a^2}=a^{-2}$; $a^{-1}=\frac{a^{-1}}{1}=\frac{1}{a}$; $\frac{1}{a^{-1}}=a^1$.

From this, we see that the reciprocal of a is $\frac{1}{a}$, or a^{-1}; of a^2 is $\frac{1}{a^2}$, or a^{-2}; of a^{-1} is $\frac{1}{a^{-1}}$, or a^1. Hence, we may obtain the reciprocal of any quantity by merely changing the sign of its exponent.

PROPOSITION

(102.) 8. *Any quantity which has zero for an exponent is equal to unity.*

DEMONSTRATION.

If we prove that $a^0=1$, we shall prove the proposition; since a may represent any quantity whatever.

We know that $\frac{a}{a}=\frac{a^1}{a^1}=1$. But, by Prop. 7, **(101.)**, $\frac{a^1}{a^1}=a^{1-1}=a^0$. Since, then, a^0 and 1 are each equal to $\frac{a^1}{a^1}$, they are equal to each other, that is, $a^0=1$.

GENERAL RULE.

(103.) *In division, coefficients are divided, and exponents subtracted.*

CASE I.

(104.) When both dividend and divisor are monomials.

RULE.

Divide according to the principles of the preceding propositions.

EXAMPLES.

1. Divide abc by ac. *Ans.* b.

2. Divide $6abc$ by $-2a$. *Ans.* $-3bc$.

3. Divide $-10xyz$ by $5y$. *Ans.* $-2xz$.

4. Divide $18ax^2$ by $3ax$. *Ans.* $6x$.

5. Divide $-28x^2y^3$ by $-4xy$. *Ans.* $7xy^2$.

6. Divide a^m by a^n. *Ans.* a^{m-n}.

7. Divide a^m by a^{-n}. *Ans.* a^{m+n}.

8. Divide a^{-m} by a^n. *Ans.* $a^{-(m+n)}$.

9. Divide a^{-m} by a^{-n}. *Ans.* a^{n-m}.

10. Divide ca^{18} by da^{-6}. *Ans.* $\frac{ca^{24}}{d}$.

11. Divide $-3a^mb^n$ by $-4a^pb^qc^r$. *Ans.* $\frac{3a^{m-p}b^{n-q}}{4c^r}$.

12. Divide $6(a+b)^{-9}$ by $4(a+b)^{-5}$. *Ans.* $\frac{3}{2(a+b)^4}$.

13. Divide $(a+x)^2(a+y)^{-3}$ by $(a+x)^{-4}(a+y)^{-7}$.
Ans. $(a+x)^6(a+y)^4$.

14. Divide $150a^5b^8cd^3$ by $30a^3b^3d^2$. *Ans.* $5a^2b^5cd$.

15. Divide $15a^{2m}x^{3n}y^{4n}$ by $3a^mx^{2n}y^{5n}$. *Ans.* $5a^mx^ny^{-n}$.

16. Divide $-48a^mb^n$ by $6a^pb^q$. *Ans.* $-8a^{m-p}b^{n-q}$.

17. Divide $6a^{\frac{2}{3}}d^{-\frac{1}{3}}$ by $3a^{-\frac{1}{3}}d^{\frac{2}{3}}$. *Ans.* $\frac{2a}{d}$.

18. Divide $12a^{-\frac{7}{8}}d^{\frac{4}{5}}c^{-\frac{3}{4}}$ by $3a^{\frac{9}{8}}d^{-\frac{1}{5}}c^{\frac{1}{4}}$. *Ans.* $\frac{4d}{a^2c}$.

19. Divide $(a+x)^{-\frac{3}{4}}$ by $5(a+x)^{\frac{9}{4}}$. *Ans.* $\frac{1}{5(a+x)^3}$.

20. Divide $(a+b)^{\frac{1}{2}}(x+y)^{-\frac{4}{5}}$ by $(a+b)^{-\frac{5}{2}}(x+y)^{\frac{11}{5}}$.

Ans. $\dfrac{(a+b)^3}{(x+y)^3}$.

CASE II.

(105.) When the dividend is a polynomial, and the divisor a monomial.

RULE.

Divide each term of the dividend by the divisor, and connect the quotients by their proper signs.

PROBLEM.

Divide $6a^3b^4-8a^2b^3d^2+4a^4b^2c$ by $2a^2b^2$.

SOLUTION.

Operation.

$$\begin{array}{r|l} 2a^2b^2) & 6a^3b^4-8a^2b^3d^2+4a^4b^2c \\ \hline & 3ab^2 \quad -4bd^2 \quad +2a^2c \end{array}$$

Dividing $6a^3b^4$, $-8a^2b^3d^2$, and $+4a^4b^2c$ respectively, by $2a^2b^2$, gives $3ab^2$, $4bd^2$, and $+2a^2c$, which, connected by their proper signs, is $3ab^2-4bd^2+2a^2c$.

EXAMPLES.

1. Divide $12a^2x+4ax^2-16a$ by $4a$. *Ans.* $3ax+x^2-4$.

2. Divide $12a^4y^6-16a^5y^5+20a^6y^4-28a^7y^3$ by $-4a^4y^3$.
Ans. $-3y^3+4ay^2-5a^2y+7a^3$.

3. Divide $15a^2bc-20acy^2+5cd^2$ by $-5abc$.
Ans. $-3a+\dfrac{4y^2}{b}-\dfrac{d^2}{ab}$.

4. Divide $x^{n+1}-x^{n+2}+x^{n+3}-x^{n+4}$ by x^n. *Ans.* $x-x^2+x^3-x^4$.

5. Divide $a^{n+1}x-a^{n+2}x-a^{n+3}x-a^{n+4}x$ by a^n.
Ans. $ax-a^2x-a^3x-a^4x$.

6. Divide $ax^n+ax^{n+1}+ax^{n+2}+ax^{n+3}$ by x^n.
Ans. $a+ax+ax^2+ax^3$.

7. Divide $6(x+y)^3-8(x+y)^2+4a^2(x+y)$ by $2(x+y)$.
Ans. $3(x+y)^2-4(x+y)+2a^2$.

8. Divide $5(a+b)^3-10(a+b)^2+15(a+b)$ by $-5(a+b)$.
Ans. $-(a+b)^2+2(a+b)-3$.

9. Divide $ax^{m-1}+bx^{m+1}-cx^{m-3}+dx^5$ by x^{m-6}.

Ans. $ax^5+bx^7-cx^3+dx^{11-m}$.

10. Divide $\frac{3}{4}a^2x^5-\frac{\ }{9}ax^3+3ab^2x$ by $\frac{2}{3}a^2x^3$.

Ans. $\frac{9}{8}x^2-\frac{7}{6a}+\frac{9b^2}{2ax^2}$.

CASE III.

(106.) When both dividend and divisor are polynomials.

RULE.

1. *Arrange both dividend and divisor according to the ascending or descending powers of the same letter in both.*

2. *Divide the first term of the dividend by the first term of the divisor; the result will be the first term of the quotient, by which multiply all the terms in the divisor, and subtract the product from the dividend.*

3. *Then to the remainder annex as many of the remaining terms of the dividend as are necessary, and find the next term of the quotient as before, and so on.*

PROBLEM

1. Divide $6a^2x^2+a^4-4a^3x+x^4-4ax^3$ by x^2+a^2-2ax.

SOLUTION.

Arranging the terms according to the descending powers of a, we have

$$
\begin{array}{l}
a^2-2ax+x^2)\,a^4-4a^3x+6a^2x^2-4ax^3+x^4\,(a^2-2ax+x^2 \\
\qquad\qquad\quad \underline{a^4-2a^3x+\ a^2x^2} \\
\qquad\qquad\qquad -2a^3x+5a^2x^2-4ax^3 \\
\qquad\qquad\qquad \underline{-2a^3x+4a^2x^2-2ax^3} \\
\qquad\qquad\qquad\qquad\qquad a^2x^2-2ax^3+x^4 \\
\qquad\qquad\qquad\qquad\qquad \underline{a^2x^2-2ax^3+x^4} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0
\end{array}
$$

ANOTHER SOLUTION.

Arranging the terms according to the descending powers of x, we have

$$
\begin{array}{l|l}
x^4-4x^3a+6x^2a^2-4xa^3+a^4 & x^2-2xa+a^2 \\
\underline{x^4-2x^3a+\ \ x^2a^2} & \overline{x^2-2xa+a^2} \\
\quad -2x^3a+5x^2a^2-4xa^3 & \\
\quad \underline{-2x^3a+4x^2a^2-2xa^3} & \\
\qquad\qquad x^2a^2-2xa^3+a^4 & \\
\qquad\qquad \underline{x^2a^2-2xa^3+a^4} & \\
\qquad\qquad\qquad\qquad 0 &
\end{array}
$$

PROBLEM.

2. Divide $2a^{3n}-6a^{2n}b^n+6a^nb^{2n}-2b^{3n}$ by a^n-b^n.

SOLUTION.

$$
\begin{array}{l}
a^n-b^n)2a^{3n}-6a^{2n}b^n+6a^nb^{2n}-2b^{3n}(2a^{2n}-4a^nb^n+2b^{2n} \\
\qquad\quad \underline{2a^{3n}-2a^{2n}b^n} \\
\qquad\qquad\quad -4a^{2n}b^n+6a^nb^{2n} \\
\qquad\qquad\quad \underline{-4a^{2n}b^n+4a^nb^{2n}} \\
\qquad\qquad\qquad\qquad 2a^nb^{2n}-2b^{3n} \\
\qquad\qquad\qquad\qquad \underline{2a^nb^{2n}-2b^{3n}} \\
\qquad\qquad\qquad\qquad\qquad\quad 0
\end{array}
$$

PROBLEM

3. Divide $x^3-(a+b+c)x^2+(ab+ac+bc)x-abc$ by $x-c$.

SOLUTION.

$$
\begin{array}{l|l}
x^3-(a+b+c)x^2+(ab+ac+bc)x-abc & x-c \\
 & \overline{x^2-(a+b)x+ab} \\
\underline{x^3 \qquad\quad -cx^2} & \\
\quad -(a+b)x^2+(ab+ac+bc)x & \\
\quad \underline{-(a+b)x^2+\qquad (ac+bc)x} & \\
\qquad\qquad\qquad abx \quad - \quad abc & \\
\qquad\qquad\qquad \underline{abx \quad - \quad abc} & \\
\qquad\qquad\qquad\qquad\qquad 0 &
\end{array}
$$

EXAMPLES.

1. Divide $a^2+2ab+b^2$ by $a+b$. *Ans.* $a+b$.

2. Divide $a^2-2ab+b^2$ by $a-b$. *Ans.* $a-b$.

3. Divide $x^4+4x^3y+6x^2y^2+4xy^3+y^4$ by $x^2+2xy+y^2$.
Ans. $x^2+2xy+y^2$.

4. Divide $x^4-4x^3y+6x^2y^2-4xy^3+y^4$ by $x-y$.
Ans. $x^3-3x^2y+3xy^2-y^3$.

5. Divide x^5-y^5 by $x-y$. *Ans.* $x^4+x^3y+x^2y^2+xy^3+y^4$.

6. Divide x^5+y^5 by $x+y$. *Ans.* $x^4-x^3y+x^2y^2-xy^3+y^4$.

7. Divide $x^3-9x^2+27x-27$ by $x-3$. *Ans.* x^2-6x+9.

8. Divide $12x^4-192$ by $3x-6$. *Ans.* $4x^3+8x^2+16x+32$.

9. Divide $6x^6-6y^6$ by $2x^2-2y^2$. *Ans.* $3x^4+3x^2y^2+3y^4$.

10. Divide $x^3+5x^2y+5xy^2+y^3$ by $x^2+4xy+y^2$. *Ans.* $x+y$.

11. Divide $a^6-3a^4b^2+3a^2b^4-b^6$ by $a^3-3a^2b+3ab^2-b^3$.
Ans. $a^3+3a^2b+3ab^2+b^3$.

12. Divide a^4-b^4 by $a^3+a^2b+ab^2+b^3$. *Ans.* $a-b$.

13. Divide $\frac{1}{2}x^3+x^2+\frac{3}{8}x+\frac{3}{4}$ by $\frac{1}{2}x+1$. *Ans.* $x^2+\frac{3}{4}$.

14. Divide x^4+y^4 by $x+y$.

Ans. $x^3-x^2y+xy^2-y^3+\dfrac{2y^4}{x+y}$.

15. Divide $x^{m+1}+x^my+xy^m+y^{m+1}$ by x^m+y^m. *Ans.* $x+y$.

16. Divide $x^{4n}+x^{2n}y^{2n}+y^{4n}$ by $x^{2n}+x^ny^n+y^{2n}$.
Ans. $x^{2n}-x^ny^n+y^{2n}$.

17. Divide $a^{m+n}b^n-4a^{m+n-1}b^{2n}-27a^{m+n-2}b^{3n}+42a^{m+n-3}b^{4n}$ by $a^nb^n-7a^{n-1}b^{2n}$. *Ans.* $a^m+3a^{m-1}b^n-6a^{m-2}b^{2n}$.

18. Divide a^n-x^n by $a-x$.

Ans. $a^{n-1}+a^{n-2}x+a^{n-3}x^2+\dfrac{a^{n-3}x^3-x^n}{a-x}$.

19. Divide $x^3-(a+b+d)x^2+(ad+bd+c)x-cd$ by $x^2-(a+b)x+c$. *Ans.* $x-d$.

20. Divide $-a^8b^4+15a^{11}b^5-48a^{14}b^6-20a^{17}b^7$ by $10a^9b^2-a^6b$.
Ans. $a^2b^3-5a^5b^4-2a^8b^5$.

21. Divide $4c^4-9b^2c^2+6b^3c-b^4$ by $2c^2-3bc+b^2$.
Ans. $2c^2+3bc-b^2$.

22. Divide $\frac{3}{4}x^5-4x^4+\frac{77}{8}x^3-\frac{43}{4}x^2-\frac{33}{4}x+27$ by $\frac{1}{2}x^2-x+3$.
Ans. $\frac{3}{2}x^3-5x^2+\frac{1}{4}x+9$.

23. Divide $-1+a^3n^3$ by $-1+an$. *Ans.* $1+an+a^2n^2$.

24. Divide $a^3d^3-3a^2cd^3+3ac^2d^3-c^3d^3+a^2c^2d^2-ac^3d^2$ by $a^2d^2-2acd^2+c^2d^2+ac^2d$. *Ans.* $ad-cd$.

25. Divide $\frac{1}{3}-6z^2+27z^4$ by $\frac{1}{3}+2z+3z^2$. *Ans.* $1-6z+9z^2$.

26. Divide $-2a^{-8}x^5+17a^{-4}x^6-5x^7-24a^4x^8$ by $2a^{-3}x^3-3ax^4$.
Ans. $-a^{-5}x^2+7a^{-1}x^3+8a^3x^4$.

27. Divide $a^{3m-2n}b^{2p}c-a^{2m+n-1}b^{1-p}c^n+a^{-n}b^{-1}c^m+a^{3m-n}b^{3p+2}c^n-a^{2m+2n-1}b^3c^{2n-1}+b^{p+1}c^{m+n-1}$ by $a^{-n}b^{-p-1}+bc^{n-1}$.

Ans. $a^{3m-n}b^{3p+1}c-a^{2m+2n-1}b^2c^n+b^pc^m$.

28. Divide $4(3a^{\frac{13}{15}}b^{-\frac{5}{6}}+7a^{-\frac{1}{20}}b^{\frac{4}{15}})c^4-3(3a^{\frac{11}{3}}b^{-\frac{11}{8}}+7a^{\frac{11}{4}}b^{-\frac{11}{40}})d^{-3}$ by $3a^{\frac{2}{3}}b^{-\frac{1}{2}}+7a^{-\frac{1}{4}}b^{\frac{3}{5}}$. *Ans.* $4a^{\frac{1}{5}}b^{-\frac{1}{3}}c^4-3a^3b^{-\frac{7}{8}}d^{-3}$.

DIVISION BY DETACHED COEFFICIENTS.

PROBLEM.

(107.) To divide by means of detached coefficients.

RULE.

Arrange the terms of the divisor and dividend according to the ascending or descending powers of a letter common to both; then, omitting the letters, write the coefficients with their respective signs, supplying the coefficients of the absent terms with zeroes. Proceeding as usual in division, the result will be the coefficients in the quotient, to which annexing the letters according to the law in each particular case, will give the complete quotient.

PROBLEM.

Divide $6a^4-96$ by $3a-6$.

SOLUTION.

Arranging the coefficients as directed and dividing, we obtain

Operation.

$$\begin{array}{l|l} 6+0+0+0-96 & 3-6 \\ \hline & 2+4+8+16 \end{array}$$

$$\begin{array}{l} 6-12 \\ \hline \quad 12+\ 0 \\ \quad 12-24 \\ \hline \qquad 24-\ 0 \\ \qquad 24-48 \\ \hline \qquad\quad 48-96 \\ \qquad\quad 48-96 \\ \hline \qquad\qquad\ 0 \end{array}$$

for the coefficients in the quotient, $+2$, $+4$, $+8$, and $+16$. An inspection of the problem shows that the first term of the divisor should contain a^3. Therefore, commencing with a^3 and inserting the descending powers of a, we have for the complete quotient, $2a^3+4a^2+8a+16$.

EXAMPLES.

1. Divide $x^4-3ax^3-8a^2x^2+18a^3x-8a^4$ by $x^2+2ax-2a^2$.
Ans. $x^2-5ax+4a^2$.

2. Divide $3y^3+3xy^2-4x^2y-4x^3$ by $x+y$. *Ans.* $-4x^2+3y^2$.

3. Divide $10a^4-27a^3x+34a^2x^2-18ax^3-8x^4$ by $2a^2-3ax+4x^2$.
Ans. $5a^2-6ax-2x^2$.

4. Divide $a^6+4a^5-8a^4-25a^3+35a^2+21a-28$ by a^2+5a+4.
Ans. $a^4-a^3-7a^2+14a-7$.

5. Divide $x^{m+1}+xy^m-x^my-y^{m+1}$ by x^m+y^m. *Ans.* $x-y$.

SYNTHETIC DIVISION.

PROBLEM.

(108.) To divide by synthetic division.

RULE.*

1. *Divide the divisor and dividend by the coefficient of the first term in the divisor, which will make the leading coefficient of the divisor unity, and the first term of the quotient will be identical with that of the dividend.*

2. *Change all the signs of the terms in the divisor, except the first, and multiply all the terms so changed by the term in the quotient, and*

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* This rule is due to Mr. W. G. Horner of Bath, England.

place the products successively under the corresponding terms of the dividend, in a diagonal column, beginning at the upper line.

3. *Add the results in the second column, which will give the second term of the quotient; and multiply the changed terms in the divisor by this result, placing the products in a diagonal series as before.*

4. *Add the results in the third column, which will give the next term in the quotient, and multiply the chanyed terms in the divisor by this term in the quotient, placing the products as before.*

5. *This process continued until the results become* 0, *or until the quotient is determined as far as necessary, will give the same series of terms as the usual mode of division when carried to an equivalent extent.*

REMARK.—In synthetic division, as in division by detached coefficients, it is customary to omit the letters.

PROBLEM.

Divide $x^6-5x^5+15x^4-24x^3+27x^2-13x+5$ by $x^4-2x^3+4x^2-2x+1$.

SOLUTION.

Since, in this problem the coefficient of the first term of the divisor is unity, Part 1st of the rule is unnecessary.

(Part 2d of the Rule.) Omitting the letters, arrange the dividend horizontally, and the divisor vertically, changing the signs of all its terms except the first.

Operation.

$$
\begin{array}{r|rrrrrrr}
1 & 1 & -5 & +15 & -24 & +27 & -13 & +5 \\
+2 & & +2 & -6 & +10 & & & \\
-4 & & & -4 & +12 & -20 & & \\
+2 & & & & +2 & -6 & +10 & \\
-1 & & & & & -1 & +3 & -5 \\
\hline
 & 1 & -3 & +5 & +0 & +0 & +0 & +0
\end{array}
$$

Then multiply the changed terms in the divisor by 1, and place the product, $+2$, -4, $+2$, and -1 diagonally under -5, $+15$, -24, and $+27$ respectively.

(Part 3d of the Rule.) Adding the results in the second column, gives -3 for the second term of the quotient, which multiplying into the changed terms of the divisor, gives -6, $+12$, -6, and $+3$, which must be placed diagonally under $+15$, -24, $+27$, and -13 respectively.

(Part 4.) Adding the terms in the third column gives $+5$ for the third term of the quotient, which multiplying into the changed terms of the divisor, gives $+10$, -20, $+10$, and -5, which must be placed

diagonally under -24, $+27$, -13, and $+5$ respectively. Here the process terminates, since the sum of each of the remaining columns equals 0.

Restoring the letters according to the law of the case, we obtain for the quotient sought x^2-3x+5.

EXAMPLES.

1. Divide $a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$ by $a^2-2ax+x^2$.
Ans. $a^3-3a^2x+3ax^2-x^3$.

2. Divide $a^6-3a^4x^2+3a^2x^4-x^6$ by $a^3-3a^2x+3ax^2-x^3$.
Ans. $a^3+3a^2x+3ax^2+x^3$.

3. Divide x^7-y^7 by $x-y$.
Ans. $x^6+x^5y+x^4y^2+x^3y^3+x^2y^4+xy^5+y^6$.

4. Divide $9x^6-46x^5+95x^3+150x$ by x^2-4x-5.
Ans. $9x^4-10x^3+5x^2-30x$.

5. Divide $25x^6-x^4-2x^3-8x^2$ by $5x^3-4x^2$.
Ans. $5x^3+4x^2+3x+2$.

6. Divide $a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$ by $a^3-3a^2x+3ax^2-x^3$.
Ans. $a^2-2ax+x^2$.

CHAPTER III.

THEOREMS AND FACTORING.

THEOREM I.

(109.) *The square of the sum of two quantities is equal to the square of the first, plus twice the product of the first by the second, plus the square of the second.*

DEMONSTRATION.

Let $a+b$ represent the sum of two quantities. Squaring it, or multiplying it by itself, we have $a^2+2ab+b^2$, or $(a+b)^2=a^2+2ab+b^2$.

PROBLEM.

Square $2a^2+3b^3$.

SOLUTION.

By the theorem, we have $(2a^2)^2+2(2a^2)(3b^3)+(3b^3)^2$, which, after the operations indicated are performed, becomes $4a^4+12a^2b^3+9b^6$.

EXAMPLES.

1. Square $x+y$. *Ans.* $x^2+2xy+y^2$.

2. Square $2x+y$. *Ans.* $4x^2+4xy+y^2$.

3. Square $3x^2+4y^2$. *Ans.* $9x^4+24x^2y^2+16y^4$.

4. Square x^3+y^2. *Ans.* $x^6+2x^3y^2+y^4$.

5. Square $3a^2+4ab^3$. *Ans.* $9a^4+24a^3b^3+16a^2b^6$.

6. Square $\frac{1}{2}c^2+dn^2$. *Ans.* $\frac{1}{4}c^4+c^2dn^2+d^2n^4$.

7. Square $a^{-1}+b^{-1}$. *Ans.* $a^{-2}+2a^{-1}b^{-1}+b^{-2}$.

8. Square $a^{\frac{2}{3}}+b^{\frac{1}{2}}$. *Ans.* $a^{\frac{4}{3}}+2a^{\frac{2}{3}}b^{\frac{1}{2}}+b$.

9. Square $2a^{-\frac{1}{2}}+3a^{-\frac{1}{2}}b^{-\frac{1}{3}}$. *Ans.* $\frac{4}{a}+\frac{12}{ab^{\frac{1}{3}}}+\frac{9}{ab^{\frac{2}{3}}}$.

10. Square $3x^{-\frac{1}{2}}y^{\frac{2}{3}}+2x^{\frac{1}{2}}y^{-\frac{2}{3}}$. *Ans.* $\frac{9y^{\frac{4}{3}}}{x}+12+\frac{4x}{y^{\frac{4}{3}}}$.

11. Square $\frac{1}{2}a^2bx^{-\frac{7}{8}}+\frac{2}{3}a^{-\frac{1}{2}}b^{-\frac{2}{3}}x^{-\frac{1}{8}}$. *Ans.* $\frac{a^4b^2}{4x^{\frac{7}{4}}}+\frac{2a^{\frac{3}{2}}b^{\frac{1}{3}}}{3x}+\frac{4}{9ab^{\frac{4}{3}}b^{\frac{1}{4}}}$.

12. Square $\frac{2b^{\frac{1}{2}}}{a^3}+\frac{b^{\frac{1}{2}}a^2}{3b^2}$. *Ans.* $4a^{-6}b+\frac{4}{3ab}+\frac{1}{9}a^4b^{-3}$.

THEOREM II.

(110.) *The square of the difference of two quantities is equal to the square of the first, minus twice the product of the first by the second, plus the square of the second.*

DEMONSTRATION.

Let $a-b$ represent the difference of two quantities. Squaring it, or multiplying it by itself, we have $a^2-2ab+b^2$, or $(a-b)^2=a^2-2ab+b^2$.

PROBLEM.

Square $5a^3-3b^4$.

SOLUTION.

By the theorem, we have $(5a^3)^2-2(5a^3)(3b^4)+(3b^4)^2$, which, after the operations indicated are performed, becomes $25a^6-30a^3b^4+9b^8$.

EXAMPLES.

1. Square $x-y$. *Ans.* $x^2-2xy+y^2$.

2. Square $\frac{1}{2}x^2-2xy^2$. *Ans.* $\frac{1}{4}x^4-2x^3y^2+4x^2y^4$.

3. Square $\frac{1}{3}m^3-\frac{1}{4}n^4$. *Ans.* $\frac{1}{9}m^6-\frac{1}{6}m^3n^4+\frac{1}{16}n^8$.

4. Square $5ay^{\frac{2}{3}}-6a^2y^{\frac{1}{3}}$. *Ans.* $25a^2y^{\frac{4}{3}}-60a^3y+36a^4y^{\frac{2}{3}}$.

5. Square $ab^{-1}-21a^{-2}b^2$. *Ans.* $\frac{a^2}{b^2}-\frac{42b}{a}+\frac{441b^4}{a^4}$.

6. Square $2a^4b^3-7a^{-3}b^{-2}$. *Ans.* $4a^8b^6-28ab+49a^{-6}b^{-4}$.

7. Square $5a^3b^2-4a^{\frac{3}{4}}b^{\frac{2}{3}}$. *Ans.* $25a^6b^4-40a^{\frac{15}{4}}b^{\frac{8}{3}}+16a^{\frac{3}{2}}b^{\frac{4}{3}}$.

8. Square $3a^{-2}b^{-3}-4a^{-1}b^{-4}$.

Ans. $9a^{-4}b^{-6}-24a^{-3}b^{-7}+16a^{-2}b^{-8}$.

9. Square $5a^{\frac{3}{2}}b^{-\frac{3}{2}}-7a^{-\frac{3}{2}}b^{\frac{3}{2}}$. *Ans.* $\frac{25a^3}{b^3}-70+\frac{49b^3}{a^3}$.

10. Square $3x^{\ +1}-4x^{-m}y^n$. *Ans.* $9x^{2m+2}-24xy^n+\frac{16y^{2n}}{x^{2m}}$.

11. Square $2x^{-(m-1)}y^{\frac{n+1}{2}}-5x^{m+1}y^{-\frac{n+1}{2}}$.

Ans. $\frac{4y^{n+1}}{x^{2m-2}}-20+\frac{25x^{2m+2}}{y^{n+1}}$.

12. Square $\frac{1}{2}x^{\frac{m+n+3}{2}}y^{-\frac{m+n}{2}}-4x^{-m-n-3}y^{\frac{m+n}{4}}$.

Ans. $\frac{x^{m+n+3}}{4y^{m+n}}-\frac{4}{x^{\frac{m+n+3}{2}}y^{\frac{m+n}{4}}}+\frac{16y^{\frac{m+n}{2}}}{x^{2(m+n+3)}}$.

THEOREM III.

(111.) *The product of the sum and difference of two quantities is equal to the difference of their squares.*

DEMONSTRATION.

Let $a+b$ and $a-b$ represent respectively the sum and difference of two quantities. Multiplying $a+b$ by $a-b$, we have a^2-b^2, or $(a+b)(a-b)=a^2-b^2$.

PROBLEM.

Find the product of $2a^2+3b^3$ by $2a^2-3b^3$.

SOLUTION.

By the theorem, we have $(2a^2)^2-(3b^3)^2$, which, after the operations indicated are performed, becomes $4a^4-9b^6$.

EXAMPLES.

1. Find the product of $m+n$ and $m-n$. *Ans.* m^2-n^2.

2. Find the product of $\frac{1}{2}a^2+\frac{1}{3}b$ and $\frac{1}{2}a^2-\frac{1}{3}b$. *Ans.* $\frac{1}{4}a^4-\frac{1}{9}b^2$.

3. Find the product of $7a^2b^2+6cd$ and $7a^2b^2-6cd$.

Ans. $49a^4b^4-36c^2d^2$.

4. Find the product of $a^{-1}+b^{-1}$ and $a^{-1}-b^{-1}$. *Ans.* $\frac{1}{a^2}-\frac{1}{b^2}$.

5. Find the product of $3ab^{-1}+\frac{1}{3}a^{-1}b$ and $3ab^{-1}-\frac{1}{3}a^{-1}b$.

Ans. $\dfrac{9a^2}{b^2}-\dfrac{b^2}{9a^2}$.

6. Find the product of $m^{\frac{1}{2}}+n^{\frac{1}{2}}$ and $m^{\frac{1}{2}}-n^{\frac{1}{2}}$. *Ans.* $m-n$.

7. Find the product of $2^{\frac{1}{2}}x^{\frac{1}{2}}+3^{\frac{1}{2}}y^{\frac{1}{2}}$ and $2^{\frac{1}{2}}x^{\frac{1}{2}}-3^{\frac{1}{2}}y^{\frac{1}{2}}$.

Ans. $2x-3y$.

8. Find the product of $3a^2b^3+2a^{\frac{1}{2}}b^{\frac{3}{4}}$ and $3a^2b^3-2a^{\frac{1}{2}}b^{\frac{3}{4}}$.

Ans. $9a^4b^6-4ab^{\frac{3}{2}}$.

9. Find the product of $\frac{2}{3}a^3b^{-3}+\frac{3}{4}a^{-3}b^3$ and $\frac{2}{3}a^3b^{-3}-\frac{3}{4}a^{-3}b^3$.

Ans. $\dfrac{4a^6}{9b^6}-\dfrac{9b^6}{16a^6}$.

10. Find the product of x^m+y^m and x^m-y^m. *Ans.* $x^{2m}-y^{2m}$.

11. Find the product of $2a^{m+1}+\frac{3}{5}a^{n-1}b$ and $2a^{m+1}-\frac{3}{5}a^{n-1}b$.

Ans. $4a^{2m+2}-\dfrac{9a^{2n-2}b^2}{25}$.

12. Find the product of $\frac{2}{3}a^{\frac{m-3}{2}}b^{\frac{3-m}{2}}+\frac{3}{2}a^{\frac{3-m}{2}}b^{\frac{m-3}{2}}$ and $\frac{2}{3}a^{\frac{m-3}{2}}b^{\frac{3-m}{2}}$ $-\frac{3}{2}a^{\frac{3-m}{2}}b^{\frac{m-3}{2}}$.

Ans. $\dfrac{4a^{m-3}}{9b^{m-3}}-\dfrac{9b^{m-3}}{4a^{m-3}}$.

THEOREM IV.

(112.) *The difference of two quantities is divisible by the difference of the same roots of the quantities.*

DEMONSTRATION.

Let $x^{\frac{m}{n}}-y^{\frac{t}{s}}$ be a general expression for the difference of two quantities, and $(x^{\frac{m}{n}})^{\frac{1}{r}}-(y^{\frac{t}{s}})^{\frac{1}{r}}$, or $x^{\frac{m}{rn}}-y^{\frac{t}{rs}}$ be a general expression for the difference of the same roots of the two quantities.

We are to prove that $x^{\frac{m}{n}}-y^{\frac{t}{s}}$ is divisible by $x^{\frac{m}{rn}}-y^{\frac{t}{rs}}$.

Dividing $x^{\frac{m}{n}}-y^{\frac{t}{s}}$ by $x^{\frac{m}{rn}}-y^{\frac{t}{rs}}$, we obtain for the

1st remainder		$x^{\frac{m}{n}-\frac{m}{rn}}y^{\frac{t}{rs}}-y^{\frac{t}{s}}$.
2d	"	$x^{\frac{m}{n}-\frac{2m}{rn}}y^{\frac{2t}{rs}}-y^{\frac{t}{s}}$.
3d	"	$x^{\frac{m}{n}-\frac{3m}{rn}}y^{\frac{3t}{rs}}-y^{\frac{t}{s}}$.
		$\vdots$
rth	"	$x^{\frac{m}{n}-\frac{rm}{rn}}y^{\frac{rt}{rs}}-y^{\frac{t}{s}}$.

Since, $\frac{rm}{rn}=\frac{m}{n}$, and $\frac{rt}{rs}=\frac{t}{s}$, this remainder $=x^{\frac{m}{n}-\frac{m}{n}}y^{\frac{t}{s}}-y^{\frac{t}{s}}=x^0y^{\frac{t}{s}}-y^{\frac{t}{s}}$. Because by Prop. 8, (**102.**) $x^0=1$, we have $x^0y^{\frac{t}{s}}-y^{\frac{t}{s}}=y^{\frac{t}{s}}-y^{\frac{t}{s}}=0$.

Hence, $x^{\frac{m}{n}}-y^{\frac{t}{s}}$ is divisible by $x^{\frac{m}{rn}}-y^{\frac{t}{rs}}$, because, after obtaining r terms in the quotient, the remainder equals zero. The form of the quotient, as we see by division, is $x^{\frac{m}{n}-\frac{m}{rn}}+x^{\frac{m}{n}-\frac{2m}{rn}}y^{\frac{t}{rs}}+x^{\frac{m}{n}-\frac{3m}{rs}}y^{\frac{2t}{rs}}$ .... $x^{\frac{m}{rn}}y^{\frac{t}{s}-\frac{2t}{rs}}+y^{\frac{t}{s}-\frac{t}{rs}}$.

Hence, $\dfrac{x^{\frac{m}{n}}-y^{\frac{t}{s}}}{x^{\frac{m}{rn}}-y^{\frac{t}{rs}}}=x^{\frac{m}{n}-\frac{m}{rn}}+x^{\frac{m}{n}-\frac{2m}{rn}}y^{\frac{t}{rs}}+x^{\frac{m}{n}-\frac{3m}{rn}}y^{\frac{2t}{rs}}$ .... $x^{\frac{2m}{rn}}y^{\frac{t}{s}-\frac{3t}{rs}}$ $+x^{\frac{m}{rn}}y^{\frac{t}{s}-\frac{2t}{rs}}+y^{\frac{t}{s}-\frac{t}{rs}}$ (A.)

When $\frac{m}{n}=u$, and $\frac{t}{s}=u$, and $r=u$, we have $\dfrac{x^u-y^u}{x-y}=x^{u-1}+x^{u-2}y+x^{u-3}y^2$ .... $x^2y^{u-3}+xy^{u-2}+y^{u-1}$. (B.)

PROBLEM

1. Divide x^5-y^5 by $x-y$.

SOLUTION.

Making, in formula (B.), $n=5$, we have $\dfrac{x^5-y^5}{x-y}=x^4+x^3y+x^2y^2+xy^3+y^4$.

PROBLEM

2. Divide $a^{\frac{9}{2}}-b^{\frac{7}{3}}$ by $\left(a^{\frac{9}{2}}\right)^{\frac{1}{5}}-\left(b^{\frac{7}{3}}\right)^{\frac{1}{5}}$, or $a^{\frac{9}{10}}-b^{\frac{7}{15}}$.

SOLUTION.

Making, in formula (A.), $\frac{m}{n}=\frac{9}{2}$, $\frac{t}{s}=\frac{7}{3}$, and $r=5$, we get $\frac{m}{rn}=\frac{9}{10}$ and $\frac{t}{rs}=\frac{7}{15}$. Also, putting $x=a$ and $y=b$, we have $\dfrac{a^{\frac{9}{2}}-b^{\frac{7}{3}}}{a^{\frac{9}{10}}-b^{\frac{7}{15}}}=a^{\frac{9}{2}-\frac{9}{10}}+a^{\frac{9}{2}-\frac{18}{10}}b^{\frac{7}{15}}+a^{\frac{9}{2}-\frac{27}{10}}b^{\frac{14}{15}}+a^{\frac{9}{2}-\frac{36}{10}}b^{\frac{21}{15}}+b^{\frac{28}{15}}=a^{\frac{36}{10}}+a^{\frac{27}{10}}b^{\frac{7}{15}}+a^{\frac{18}{10}}b^{\frac{14}{15}}+a^{\frac{9}{10}}b^{\frac{21}{15}}+b^{\frac{28}{15}}=a^{\frac{18}{5}}+a^{\frac{27}{10}}b^{\frac{7}{15}}+a^{\frac{9}{5}}b^{\frac{14}{15}}+a^{\frac{9}{10}}b^{\frac{7}{5}}+b^{\frac{28}{15}}$.

EXAMPLES.

1. Divide x^3-a^3 by $x-a$. *Ans.* x^2+ax+a^2.

2. Divide a^5-b^5 by $a-b$. *Ans.* $a^4+a^3b+a^2b^2+ab^3+b^4$.

3. Divide a^6-b^6 by $a-b$.
Ans. $a^5+a^4b+a^3b^2+a^2b^3+ab^4+b^5$.

4. Divide a^7-b^7 by $a-b$.
Ans. $a^6+a^5b+a^4b^2+a^3b^3+a^2b^4+ab^5+b^6$.

5. Divide a^8-b^8 by $a-b$.
Ans. $a^7+a^6b+a^5b^2+a^4b^3+a^3b^4+a^2b^5+ab^6+b^7$.

6. Divide x^2-y^2 by $x^{\frac{1}{2}}-y^{\frac{1}{2}}$. *Ans.* $x^{\frac{3}{2}}+xy^{\frac{1}{2}}+x^{\frac{1}{2}}y+y^{\frac{3}{2}}$.

7. Divide x^2-y^2 by $x^{\frac{2}{5}}-y^{\frac{2}{5}}$.
Ans. $x^{\frac{8}{5}}+x^{\frac{6}{5}}y^{\frac{2}{5}}+x^{\frac{4}{5}}y^{\frac{4}{5}}+x^{\frac{2}{5}}y^{\frac{6}{5}}+y^{\frac{8}{5}}$.

8. Divide x^5-y^5 by $x^{\frac{5}{6}}-y^{\frac{5}{6}}$.
Ans. $x^{\frac{25}{6}}+x^{\frac{10}{3}}y^{\frac{5}{6}}+x^{\frac{5}{2}}y^{\frac{5}{3}}+x^{\frac{5}{3}}y^{\frac{5}{2}}+x^{\frac{5}{6}}y^{\frac{10}{3}}+y^{\frac{25}{6}}$.

9. Divide x^2-y by $x^{\frac{1}{2}}-y^{\frac{1}{4}}$. *Ans.* $x^{\frac{3}{2}}+xy^{\frac{1}{4}}+x^{\frac{1}{2}}y^{\frac{1}{2}}+y^{\frac{3}{4}}$.

10. Divide $m^{\frac{2}{3}}-n^{\frac{7}{8}}$ by $m^{\frac{2}{21}}-n^{\frac{1}{8}}$.
Ans. $m^{\frac{4}{7}}+m^{\frac{10}{21}}n^{\frac{1}{8}}+m^{\frac{8}{21}}y^{\frac{1}{4}}+m^{\frac{2}{7}}y^{\frac{3}{8}}+m^{\frac{4}{21}}y^{\frac{1}{2}}+m^{\frac{2}{21}}y^{\frac{5}{8}}+y^{\frac{3}{4}}$.

11. Divide x^3-y^{-3} by $x^{\frac{3}{4}}-y^{-\frac{3}{4}}$.
Ans. $x^{\frac{9}{4}}+x^{\frac{3}{2}}y^{-\frac{3}{4}}+x^{\frac{3}{4}}y^{-\frac{3}{2}}+y^{-\frac{9}{4}}$.

12. Divide $a^{-\frac{4}{5}}-x^{\frac{4}{7}}$ by $a^{-\frac{4}{25}}-x^{\frac{4}{35}}$.
Ans. $a^{-\frac{16}{25}}+a^{-\frac{12}{25}}x^{\frac{4}{35}}+a^{-\frac{8}{25}}x^{\frac{8}{35}}+a^{-\frac{4}{25}}x^{\frac{12}{35}}+x^{\frac{16}{35}}$.

THEOREM V.

(113.) *The difference of two quantities is divisible by the sum of the same roots of the quantities, when the index of the root is even.*

DEMONSTRATION.

Let $x^{\frac{m}{n}}-y^{\frac{t}{s}}$ be a general expression for the difference of two quantities, and $x^{\left(\frac{m}{n}\right)\frac{1}{r}}+\left(y^{\frac{t}{s}}\right)^{\frac{1}{r}}$, or $x^{\frac{m}{rn}}+y^{\frac{t}{rs}}$ be a general expression for the sum of the same roots of the two quantities. We are to prove that $x^{\frac{m}{n}}-y^{\frac{t}{s}}$ is divisible by $x^{\frac{m}{rn}}+y^{\frac{t}{rs}}$, when r is an even number.

Dividing $x^{\frac{m}{n}}-y^{\frac{t}{s}}$ by $x^{\frac{m}{rn}}+y^{\frac{t}{rs}}$, we obtain for the

1st remainder $\quad -x^{\frac{m}{n}-\frac{m}{rn}}y^{\frac{t}{rs}}-y^{\frac{t}{s}}$

2d " $\quad x^{\frac{m}{n}-\frac{2m}{rn}}y^{\frac{2t}{rs}}-y^{\frac{t}{s}}$

3d " $\quad -x^{\frac{m}{n}-\frac{3m}{rn}}y^{\frac{3t}{rs}}-y^{\frac{t}{s}}$

4th " $\quad x^{\frac{m}{n}-\frac{4m}{rn}}y^{\frac{4t}{rs}}-y^{\frac{t}{s}}$

$\vdots \qquad\qquad \vdots \qquad \vdots$

rth remainder, $\quad x^{\frac{m}{n}-\frac{rm}{rn}}y^{\frac{rt}{rs}}-y^{\frac{t}{s}}$ when r is an even number.

This remainder $=0$ as was shown in the *Dem.* of *Theorem 4th.* Hence, $x^{\frac{m}{n}}-y^{\frac{t}{s}}$ is divisible by $x^{\frac{m}{rn}}+y^{\frac{t}{rs}}$, when r is an even number, because after obtaining r terms in the quotient, the remainder equals zero.

The form of the quotient, as we see by division, is, $x^{\frac{m}{n}-\frac{m}{rn}}-x^{\frac{m}{n}-\frac{2m}{rn}}y^{\frac{t}{rs}}+x^{\frac{m}{n}-\frac{3m}{rn}}y^{\frac{2t}{rs}}-x^{\frac{m}{n}-\frac{4m}{rn}}y^{\frac{3t}{rs}}+\dots\dots\dots x^{\frac{m}{rn}}y^{\frac{t}{s}-\frac{2t}{rs}}-y^{\frac{t}{s}-\frac{t}{rs}}$.

Hence, when r is an even number, $\dfrac{x^{\frac{m}{n}}-y^{\frac{t}{s}}}{x^{\frac{m}{rn}}+y^{\frac{t}{rs}}}=x^{\frac{m}{n}-\frac{m}{rn}}-x^{\frac{m}{n}-\frac{2m}{rn}}y^{\frac{t}{rs}}+$

$x^{\frac{m}{n}-\frac{3m}{rn}}y^{\frac{2t}{rs}}-x^{\frac{m}{n}-\frac{4m}{rn}}y^{\frac{3t}{rs}}+\dots\dots\dots\dots\dots\dots x^{\frac{3m}{rn}}y^{\frac{t}{s}-\frac{4t}{rs}}-x^{\frac{2m}{rn}}y^{\frac{t}{s}-\frac{3t}{rs}}$
$+x^{\frac{m}{rn}}y^{\frac{t}{s}-\frac{2t}{rs}}y^{\frac{t}{s}-\frac{t}{rs}}$. (A.)

When $\frac{m}{n}=u$; $\frac{t}{s}=u$; and $r=u$, r being an even number, we have

$\dfrac{x^u-y^u}{x-y}=x^{u-1}-x^{u-2}y+x^{u-3}y^2-x^{u-4}y^3+\dots\dots\dots\dots-x^2y^{u-3}+xy^{u-2}$
$-y^{n-1}$. (B.)

PROBLEM

1. Divide a^6-b^6 by $a+b$.

SOLUTION.

Making, in formula (B.), $u=6$, we have

$$\frac{a^6-b^6}{a+b}=a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5.$$

PROBLEM

2. Divide a^5-b^5 by $a^{\frac{5}{6}}+b^{\frac{5}{6}}$.

SOLUTION.

Making, in formula (A.), $\frac{m}{n}=5, \frac{t}{s}=5$, and $r=6$, we get $\frac{m}{rn}=\frac{5}{6}$ and $\frac{t}{rs}=\frac{5}{6}$.

Also, putting $x=a$, and $y=b$, we have

$$\frac{a^5-b^5}{a^{\frac{5}{6}}+b^{\frac{5}{6}}}=a^{5-\frac{5}{6}}-a^{5-\frac{10}{6}}b^{\frac{5}{6}}+a^{5-\frac{15}{6}}b^{\frac{10}{6}}-a^{5-\frac{20}{6}}b^{\frac{15}{6}}+a^{5-\frac{25}{6}}b^{\frac{20}{6}}-b^{\frac{25}{6}}=$$

$$a^{\frac{25}{6}}-a^{\frac{10}{3}}b^{\frac{5}{6}}+a^{\frac{5}{2}}b^{\frac{5}{3}}-a^{\frac{5}{3}}b^{\frac{5}{2}}+a^{\frac{5}{6}}b^{\frac{10}{3}}-b^{\frac{25}{6}}.$$

EXAMPLES.

1. Divide a^4-b^4 by $a+b$. *Ans.* $a^3-a^2b+ab^2-b^3$.
2. Divide a^8-b^8 by $a+b$.
 Ans. $a^7-a^6b+a^5b^2-a^4b^3+a^3b^4-a^2b^5+ab^6-b^7$.
3. Divide $a-b$ by $a^{\frac{1}{4}}+b^{\frac{1}{4}}$. *Ans.* $a^{\frac{3}{4}}-a^{\frac{1}{2}}b^{\frac{1}{4}}+a^{\frac{1}{4}}b^{\frac{1}{2}}-b^{\frac{3}{4}}$.
4. Divide a^5-b^5 by $a^{\frac{5}{4}}+b^{\frac{5}{4}}$. *Ans.* $a^{\frac{15}{4}}-a^{\frac{5}{2}}b^{\frac{5}{4}}+a^{\frac{5}{4}}b^{\frac{5}{2}}-b^{\frac{15}{4}}$.
5. Divide m^5-n^3 by $m^{\frac{5}{4}}+n^{\frac{3}{4}}$. *Ans.* $m^{\frac{15}{4}}-m^{\frac{5}{2}}n^{\frac{3}{4}}+m^{\frac{5}{4}}n^{\frac{3}{2}}-n^{\frac{9}{4}}$.
6. Divide x^4-y^7 by $x+y^{\frac{7}{4}}$. *Ans.* $x^3-x^2y^{\frac{7}{4}}+xy^{\frac{7}{2}}-y^{\frac{21}{4}}$.
7. Divide x^2-y^6 by $x^{\frac{1}{3}}+y$.
 Ans. $x^{\frac{5}{3}}-x^{\frac{4}{3}}y+xy^2+x^{\frac{2}{3}}y^3+x^{\frac{1}{3}}y^4-y^5$.
8. Divide x^7-y^3 by $x^{\frac{7}{8}}+y^{\frac{3}{8}}$.
 Ans. $x^{\frac{49}{8}}-x^{\frac{21}{4}}y^{\frac{3}{8}}+x^{\frac{35}{8}}y^{\frac{3}{4}}-x^{\frac{7}{2}}y^{\frac{9}{8}}+x^{\frac{21}{8}}y^{\frac{3}{2}}-x^{\frac{7}{4}}y^{\frac{15}{8}}+x^{\frac{7}{8}}y^{\frac{9}{4}}-y^{\frac{21}{8}}$.
9. Divide $x^{\frac{2}{3}}-y^{\frac{3}{4}}$ by $x^{\frac{1}{6}}+y^{\frac{3}{16}}$.
 Ans. $x^{\frac{1}{2}}-x^{\frac{1}{3}}y^{\frac{3}{16}}+x^{\frac{1}{6}}y^{\frac{3}{8}}-y^{\frac{9}{16}}$.
10. Divide $x^{-4}-y^{-4}$ by $x^{-1}+y^{-1}$.
 Ans. $x^{-3}-x^{-2}y^{-1}+x^{-1}y^{-2}-y^{-3}$.
11. Divide x^4-y^{-3} by $x+y^{-\frac{3}{4}}$. *Ans.* $x^3-x^2y^{-\frac{3}{4}}+xy^{-\frac{3}{2}}-y^{-\frac{9}{4}}$.
12. Divide $x^{\frac{3}{5}}-y^{-\frac{5}{7}}$ by $x^{-\frac{1}{10}}+y^{-\frac{5}{42}}$.
 Ans. $x^{\frac{1}{2}}-x^{\frac{2}{5}}y^{-\frac{5}{42}}+x^{\frac{3}{10}}y^{-\frac{5}{21}}-x^{\frac{1}{5}}y^{-\frac{5}{14}}+x^{\frac{1}{10}}y^{-\frac{10}{21}}-y^{-\frac{25}{42}}$.

THEOREM VI.

(114.) *The sum of two quantities is divisible by the sum of the same odd roots of the quantities.*

DEMONSTRATION.

Let $x^{\frac{m}{n}}+y^{\frac{t}{s}}$ be a general expression for the sum of two quantities, and $\left(x^{\frac{m}{n}}\right)^{\frac{1}{r}}+\left(y^{\frac{t}{s}}\right)^{\frac{1}{r}}$, or $x^{\frac{m}{rn}}+y^{\frac{t}{rs}}$ be a general expression for the sum of the same roots of the two quantities.

We are to prove that $x^{\frac{m}{n}}+y^{\frac{t}{s}}$ is divisible by $x^{\frac{m}{rn}}+y^{\frac{t}{rs}}$, when r is an odd number.

Dividing $x^{\frac{m}{n}}+y^{\frac{t}{s}}$ by $x^{\frac{m}{rn}}+y^{\frac{t}{rs}}$, we obtain for the

1st remainder $\quad -x^{\frac{m}{n}-\frac{m}{rn}}y^{\frac{m}{rs}}+y^{\frac{t}{s}}$.

2d " $\quad x^{\frac{m}{n}-\frac{2m}{rn}}y^{\frac{2t}{rs}}+y^{\frac{t}{s}}$.

3d " $\quad -x^{\frac{m}{n}-\frac{3m}{rn}}y^{\frac{3t}{rs}}+y^{\frac{t}{s}}$.

4th " $\quad x^{\frac{m}{n}-\frac{5m}{rn}}y^{\frac{4t}{rs}}+y^{\frac{t}{s}}$.

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rth remainder $\quad -x^{\frac{m}{n}-\frac{rm}{rn}}y^{\frac{rt}{rs}}+y^{\frac{t}{s}}$ when r is an odd number.

This remainder is obviously equal to zero. Hence, $x^{\frac{m}{n}}+y^{\frac{t}{s}}$ is divisible by $x^{\frac{m}{rn}}+y^{\frac{t}{rs}}$, when r is an odd number, because after obtaining r terms in the quotient, the remainder is zero.

The form of the quotient, as we see by division, is $x^{\frac{m}{n}-\frac{m}{rn}}-x^{\frac{m}{n}-\frac{2m}{rn}}y^{\frac{t}{rs}}+x^{\frac{m}{n}-\frac{3m}{rn}}y^{\frac{2t}{rs}}+\dots\dots\dots\dots+x^{\frac{2m}{rn}}y^{\frac{t}{s}-\frac{3t}{rs}}-x^{\frac{m}{rn}}y^{\frac{t}{s}-\frac{2t}{rs}}+y^{\frac{t}{s}-\frac{t}{rs}}$.

Hence, when r is an odd number $\dfrac{x^{\frac{m}{n}}+y^{\frac{t}{s}}}{x^{\frac{m}{rn}}+y^{\frac{t}{rs}}}=x^{\frac{m}{n}-\frac{m}{rn}}-x^{\frac{m}{n}-\frac{2m}{rn}}y^{\frac{t}{rs}}+x^{\frac{m}{n}-\frac{3m}{rn}}y^{\frac{2t}{rs}}-\dots\dots\dots\dots+x^{\frac{2m}{rn}}y^{\frac{t}{s}-\frac{3t}{rs}}-x^{\frac{m}{rn}}y^{\frac{t}{s}-\frac{2t}{rs}}+y^{\frac{t}{s}-\frac{t}{rs}}$. (A.)

When $\frac{m}{n}=u$, $\frac{t}{s}=u$, and $r=u$, r being an odd number, $\dfrac{x^u+y^u}{x+y}=x^{u-1}-x^{u-2}y+x^{u-3}y^2-x^{u-4}y^3+\dots\dots\dots+x^2y^{u-3}-xy^{u-2}+y^{u-1}$. (B).

PROBLEM

1. Divide a^5+b^5 by $a+b$.

SOLUTION.

Making, in formula (B.), $u=5$, we have $\dfrac{a^5+b^5}{a+b}=a^4-a^3b+a^2b^2-ab^3+b^4$.

PROBLEM

2. Divide a^8+b^8 by $a^{\frac{8}{3}}+b^{\frac{8}{3}}$.

SOLUTION.

Making, in formula (A.), $\dfrac{m}{n}=8$, $\dfrac{t}{s}=8$, and $r=3$, we get $\dfrac{m}{rn}=\dfrac{8}{3}$, and $\dfrac{t}{rs}=\dfrac{8}{3}$.

Also, puting $x=a$, and $y=b$, we have

$$\frac{a^8+b^8}{a^{\frac{8}{3}}+b^{\frac{8}{3}}}=a^{\frac{16}{3}}-a^{\frac{8}{3}}b^{\frac{8}{3}}+b^{\frac{16}{3}}.$$

EXAMPLES.

1. Divide a^3+b^3 by $a+b$. *Ans.* a^2-ab+b^2.

2. Divide $a^{10}+b^{10}$ by a^2+b^2. *Ans.* $a^8-a^6b^2+a^4b^4-a^2b^6+b^8$.

3. Divide a^7+b^7 by $a+b$.
Ans. $a^6-a^5b+a^4b^2-a^3b^3+a^2b^4-ab^5+b^6$.

4. Divide $a+b$ by $a^{\frac{1}{5}}+b^{\frac{1}{5}}$.
Ans. $a^{\frac{4}{5}}-a^{\frac{3}{5}}b^{\frac{1}{5}}+a^{\frac{2}{5}}b^{\frac{2}{5}}+a^{\frac{1}{5}}b^{\frac{3}{5}}+b^{\frac{4}{5}}$.

5. Divide a^6+b^6 by $a^{\frac{6}{5}}+b^{\frac{6}{5}}$.
Ans. $a^{\frac{24}{5}}-a^{\frac{18}{5}}b^{\frac{6}{5}}+a^{\frac{12}{5}}b^{\frac{12}{5}}-a^{\frac{6}{5}}b^{\frac{18}{5}}+b^{\frac{24}{5}}$.

6. Divide a^6+b^5 by $a^{\frac{6}{5}}+b$.
Ans. $a^{\frac{24}{5}}-a^{\frac{18}{5}}b+a^{\frac{12}{5}}b^2-a^{\frac{6}{5}}b^3+b^4$.

7. Divide a^5+b^4 by $a+b^{\frac{4}{5}}$. *Ans.* $a^4-a^3b^{\frac{4}{5}}+a^2b^{\frac{8}{5}}-ab^{\frac{12}{5}}+b^{\frac{16}{5}}$.

8. Divide $a+b^2$ by $a^{\frac{1}{5}}+b^{\frac{2}{5}}$.
Ans. $a^{\frac{4}{5}}-a^{\frac{3}{5}}b^{\frac{2}{5}}+a^{\frac{2}{5}}b^{\frac{4}{5}}-a^{\frac{1}{5}}b^{\frac{6}{5}}+b^{\frac{8}{5}}$.

9. Divide $a^{-5}+b^{-5}$ by $a^{-1}+b^{-1}$.
Ans. $a^{-4}-a^{-3}b^{-1}+a^{-2}b^{-2}-a^{-1}b^{-3}+b^{-4}$.

10. Divide $a^{\frac{2}{3}}+b^{\frac{3}{4}}$ by $a^{\frac{2}{15}}+b^{\frac{3}{20}}$.

Ans. $a^{\frac{8}{15}}-a^{\frac{6}{15}}b^{\frac{3}{20}}+a^{\frac{4}{15}}b^{\frac{3}{10}}-a^{\frac{2}{15}}b^{\frac{9}{20}}+b^{\frac{3}{5}}$.

11. Divide $a^{\frac{2}{5}}+b^{-\frac{4}{5}}$ by $a^{\frac{2}{25}}+b^{-\frac{4}{25}}$.

Ans. $a^{\frac{8}{25}}-a^{\frac{6}{25}}b^{-\frac{4}{25}}+a^{\frac{4}{25}}b^{-\frac{8}{25}}-a^{\frac{2}{25}}b^{-\frac{12}{25}}+b^{-\frac{16}{25}}$.

THEOREM VII.

(115.) *The sum of the squares of two quantities, plus twice the product of the quantities, is equal to the square of the sum of the quantities.*

DEMONSTRATION.

Let a and b represent any two quantities, then a^2+b^2+2ab, or $a^2+2ab+b^2$ represents the sum of their squares plus twice their product.

We are to prove that a^2+b^2+2ab, or $a^2+2ab+b^2=(a+b)^2$. By Theorem 1, we have $(a+b)^2=a^2+2ab+b^2$; $\therefore$ $a^2+2ab+b^2$, or $a^2+b^2+2ab=(a+b)^2$.

PROBLEM

1. Find the factors of x^2+4x+4.

SOLUTION.

Since x^2+4x+4, or $x^2+4+4x=x^2+2^2+2\cdot 2x$, it represents the sum of the squares of x and 2, plus twice their product. Hence, by the Theorem, we have $x^2+2^2+2\cdot 2x$, or $x^2+4x+4=(x+2)^2=(x+2)(x+2)$.

EXAMPLES.

1. Find the factors of x^2+y^2+2xy. *Ans.* $(x+y)(x+y)$.

2. Find the factors of $a^2+2ay+y^2$. *Ans.* $(a+y)(a+y)$.

3. Find the factors of $x^2+10x+25$. *Ans.* $(x+5)(x+5)$.

4. Find the factors of $x^2+24x+144$. *Ans.* $(x+12)(x+12)$.

5. Find the factors of $m^2+114m+3249$.

Ans. $(m+57)(m+57)$.

6. Find the factors $x+2x^{\frac{1}{2}}y^{\frac{1}{2}}+y$. *Ans.* $(x^{\frac{1}{2}}+y^{\frac{1}{2}})(x^{\frac{1}{2}}+y^{\frac{1}{2}})$.

7. Find the factors of $4x^2+4x+1$. *Ans.* $(2x+1)(2x+1)$.

8. Find the factors of $169x^2+26x+1$. *Ans.* $(13x+1)(13x+1)$.

9. Find the factors of $25x^{-2}+60x^{-1}y^{\frac{1}{3}}+36y^{\frac{2}{3}}$.
Ans. $(5x^{-1}+6y^{\frac{1}{3}})(5x^{-1}+6y^{\frac{1}{3}})$.

10. Find the factors of $x+a+2a^{\frac{1}{2}}x^{\frac{1}{2}}$. *Ans.* $(x^{\frac{1}{2}}+a^{\frac{1}{2}})(x^{\frac{1}{2}}+a^{\frac{1}{2}})$.

11. Find the factors of $x^{\frac{3}{4}}+a^2x^{\frac{5}{4}}+2ax$.
Ans. $(x^{\frac{3}{8}}+ax^{\frac{5}{8}})(x^{\frac{3}{8}}+ax^{\frac{5}{8}})$.

12. Find the factors of $9x^{\frac{1}{2}}+49x^{\frac{3}{2}}y^{-\frac{4}{5}}+42xy^{-\frac{2}{5}}$.
Ans. $(3x^{\frac{1}{4}}+7x^{\frac{3}{4}}y^{-\frac{2}{5}})(3x^{\frac{1}{4}}+7x^{\frac{3}{4}}y^{-\frac{2}{5}})$.

THEOREM VIII.

(116.) *The sum of the squares of two quantities, minus twice the product of the quantities, is equal to the square of the difference of the quantities.*

DEMONSTRATION.

Let a and b represent any two quantities, then a^2+b^2-2ab, or $a^2-2ab+b^2$ represents the sum of their squares, minus twice their product. We are to prove that a^2+b^2-2ab, or $a^2-2ab+b^2=(a-b)^2$.

By Theorem II. we have $(a-b)^2=a^2-2ab+b^2$; $\therefore$ $a^2-2ab+b^2$, or $a^2+b^2-2ab=(a-b)^2$.

PROBLEM.

Find the factors of $9x+4y^{\frac{1}{2}}-12x^{\frac{1}{2}}y^{\frac{1}{4}}$.

SOLUTION.

Since $9x+4y^{\frac{1}{2}}-12x^{\frac{1}{2}}y^{\frac{1}{4}}=(3x^{\frac{1}{2}})^2+(2y^{\frac{1}{4}})^2-2\cdot(3x^{\frac{1}{2}})(2y^{\frac{1}{4}})$, it represents the sum of the squares of two quantities, minus twice their product. Hence, by the Theorem, we have $9x+4y^{\frac{1}{2}}-12x^{\frac{1}{2}}y^{\frac{1}{4}}$, or $9x-12x^{\frac{1}{2}}y^{\frac{1}{4}}+4y^{\frac{1}{2}}=(3x^{\frac{1}{2}}-2y^{\frac{1}{4}})^2=(3x^{\frac{1}{2}}-2y^{\frac{1}{4}})(3x^{\frac{1}{2}}-2y^{\frac{1}{4}})$.

EXAMPLES.

1. Find the factors of $x^2-2xy+y^2$. *Ans.* $(x-y)(x-y)$.

2. Find the factors of $m^2-4mn+4n^2$. *Ans.* $(m-2n)(m-2n)$.

3. Find the factors of $x^2+\frac{1}{9}-\dfrac{2x}{3}$. *Ans.* $(x-\frac{1}{3})(x-\frac{1}{3})$.

4. Find the factors of $a^3-2a^{\frac{3}{2}}x^{\frac{3}{2}}+x^3$. *Ans.* $(a^{\frac{3}{2}}-x^{\frac{3}{2}})(a^{\frac{3}{2}}-x^{\frac{3}{2}})$.

5. Find the factors of $a^5-4a^{\frac{5}{2}}y^3+4y^6$. *Ans.* $(a^{\frac{5}{2}}-2y^3)(a^{\frac{5}{2}}-2y^3)$.

6. Find the factors of $9x^8+9y-18x^4y^{\frac{1}{2}}$.

Ans. $(3x^4-3y^{\frac{1}{2}})(3x^4-3y^{\frac{1}{2}})$.

7. Find the factors of $\dfrac{9}{16}x^5-3x^{\frac{5}{2}}+4$. *Ans.* $(\frac{3}{4}x^{\frac{5}{2}}-2)(\frac{3}{4}x^{\frac{5}{2}}-2)$.

8. Find the factors of $16x^{\frac{2}{3}}-16x^{\frac{1}{3}}y^{-1}+4y^{-2}$.

Ans. $(4x^{\frac{1}{3}}-2y^{-1})(4x^{\frac{1}{3}}-2y^{-1})$.

9. Find the factors of $3x^{\frac{3}{4}}+3y^{-\frac{2}{3}}-6x^{\frac{3}{8}}y^{-\frac{1}{3}}$.

Ans. $(3^{\frac{1}{2}}x^{\frac{3}{8}}-3^{\frac{1}{2}}y^{-\frac{1}{3}})\,(3^{\frac{1}{2}}x^{\frac{3}{8}}-3^{\frac{1}{2}}y^{-\frac{1}{3}})$.

10. Find the factors of $x^{2m}+y^{2n}-2x^my^n$.

Ans. $(x^m-y^n)(x^m-y^n)$.

11. Find the factors of $4x^{4m}-16x^{2m}y^{\frac{n}{2}}+16y^n$.

Ans. $(2x^{2m}-4y^{\frac{n}{2}})(2x^{2m}-4y^{\frac{n}{2}})$.

12. Find the factors of $x^{\frac{m}{n}}y^{-\frac{t}{s}}-2+x^{-\frac{m}{n}}y^{\frac{t}{s}}$.

Ans. $(x^{\frac{m}{2n}}y^{-\frac{t}{2s}}-x^{-\frac{m}{2n}}y^{\frac{t}{2s}})(x^{\frac{m}{2n}}-x^{-\frac{m}{2n}}y^{\frac{t}{2s}})$.

THEOREM IX.

(117.) *The difference of the squares of two quantities is equal to the product of the sum and difference of the quantities.*

DEMONSTRATION.

Let a and b represent any two quantities, then a^2-b^2 represents the difference of their squares.

We are to prove that $a^2-b^2=(a+b)(a-b)$.

By Theorem III., we have $(a+b)(a-b)=a^2-b^2$), $\therefore$ $a^2-b^2=(a+b)(a-b)$.

PROBLEM.

Find the factors of $a-b$.

SOLUTION.

Since $a-b=(a^{\frac{1}{2}})^2-(b^{\frac{1}{2}})^2$, it represents the difference of the squares of the quantities $a^{\frac{1}{2}}$ and $b^{\frac{1}{2}}$.

Hence, by the Theorem, we have $a-b=(a^{\frac{1}{2}}+b^{\frac{1}{2}})(a^{\frac{1}{2}}-b^{\frac{1}{2}})$.

EXAMPLES.

1. Find the factors of x^2-y^2. *Ans.* $(x+y)(x-y)$.

2. Find the factors of x^8-y^6. *Ans.* $(x^4+y^3)(x^4-y^3)$.

3. Find the factors of x^3-y. *Ans.* $(x^{\frac{3}{2}}+y^{\frac{1}{2}})(x^{\frac{3}{2}}-y^{\frac{1}{2}})$.

4. Find the factors of x^3-y^5. *Ans.* $(x^{\frac{3}{2}}+y^{\frac{5}{2}})(x^{\frac{3}{2}}-y^{\frac{5}{2}})$.

5. Find the factors of $x^{-2}-y^{-2}$. *Ans.* $\left(\frac{1}{x}+\frac{1}{y}\right)(x^{-1}-y^{-1})$.

6. Find the factors of $9x^2-4y^4$. *Ans.* $(3x+2y^2)(3x-2y^2)$.

7. Find the factors of $2x-2y$.
Ans. $[(2x)^{\frac{1}{2}}+(27y)^{\frac{1}{2}}](2^{\frac{1}{2}}x^{\frac{1}{2}}-2^{\frac{1}{2}}y^{\frac{1}{2}})$.

8. Find the factors of $3x^{\frac{1}{3}}-5y^{\frac{1}{5}}$.
Ans. $(3^{\frac{1}{2}}x^{\frac{1}{6}}+5^{\frac{1}{2}}y^{\frac{1}{10}})(3^{\frac{1}{2}}x^{\frac{1}{6}}-5^{\frac{1}{2}}y^{\frac{1}{10}})$.

9. Find the factors of $4x^{\frac{4}{7}}-y^{-\frac{2}{3}}$. *Ans.* $\left(2x^{\frac{2}{7}}+\frac{1}{y^{\frac{1}{3}}}\right)\left(2x^{\frac{2}{7}}-\frac{1}{y^{\frac{1}{3}}}\right)$.

10. Find the factors of $16x^{-2}-25y$. *Ans.* $\left(\frac{4}{x}+5y^{\frac{1}{2}}\right)\left(\frac{4}{x}-5y^{\frac{1}{2}}\right)$.

11. Find the factors of $4x^m-9y^{-n}$.
Ans. $\left(2x^{\frac{m}{2}}+3y^{-\frac{n}{2}}\right)\left(2x^{\frac{m}{2}}-3y^{-\frac{n}{2}}\right)$.

12. Find the factors of $x^{-\frac{2m}{n}}-4x^{\frac{2m}{n}}y^{-\frac{2t}{s}}$.

Ans. $\left(\frac{1}{x^{\frac{m}{n}}}+\frac{2x^{\frac{m}{n}}}{y^{\frac{t}{s}}}\right)\left(\frac{1}{x^{\frac{m}{n}}}-\frac{2x^{\frac{m}{n}}}{y^{\frac{t}{s}}}\right)$.

THEOREM X.

(118.) The expression $x^{\frac{m}{n}}-y^{\frac{t}{s}}=\left(x^{\frac{m}{rn}}-y^{\frac{t}{rs}}\right)\left(x^{\frac{m}{n}-\frac{m}{rn}}+x^{\frac{m}{n}-\frac{2m}{rn}}y^{\frac{t}{rs}}+x^{\frac{m}{n}-\frac{3m}{rn}}y^{\frac{2t}{rs}}+\dots\dots\dots\dots+x^{\frac{2m}{rn}}y^{\frac{t}{s}-\frac{3t}{rs}}+x^{\frac{m}{rn}}y^{\frac{t}{s}-\frac{2t}{rs}}+y^{\frac{t}{s}-\frac{t}{rs}}\right)$, which, when $\frac{m}{n}$, $\frac{t}{s}$, and r respectively equals u, becomes $x^u-y^u=(x-y)(x^{u-1}+x^{u-2}y+x^{u-3}y^2+\dots\dots\dots\dots+x^2y^{u-3}+xy^{u-2}+y^{u-1}$.

DEMONSTRATION.

The truth of this Theorem depends on the truth of Theorem IV., and the fact that the divisor multiplied by the quotient equals the dividend.

PROBLEM

1. Find the factors of x^5-y^5.

SOLUTION.

In this $u=5$, and making $r=5$, we have, by the Theorem, $x^5-y^5=(x-y)(x^4+x^3y+x^2y^2+xy^3+y^4.)$

Other factors may be obtained by assigning other positive integral values to r.

PROBLEM

2. Find the factors of $x^{\frac{3}{4}}-y^{\frac{2}{3}}$.

SOLUTION.

In this example, $\frac{m}{n}=\frac{3}{4}$, and $\frac{t}{s}=\frac{2}{3}$, and we are at liberty to make r any positive integer. Let r then equal 5; whence, $\frac{m}{rn}=\frac{3}{20}$, and $\frac{t}{rs}=\frac{2}{15}$. Hence, by the Theorem, we have $x^{\frac{3}{4}}-y^{\frac{2}{3}}=\left(x^{\frac{3}{20}}-y^{\frac{2}{15}}\right)\left(x^{\frac{3}{5}}+x^{\frac{9}{20}}y^{\frac{2}{15}}+x^{\frac{3}{10}}y^{\frac{4}{15}}+x^{\frac{3}{20}}y^{\frac{2}{5}}+y^{\frac{8}{15}}\right)$. Other factors may be obtained by assigning different positive integral values to r.

EXAMPLES.

1. Find the factors of a^3-x^3.

Ans. $(a-x)(a^2+ax+x^2)$, or $\left(a^{\frac{3}{2}}-x^{\frac{3}{2}}\right)\left(a^{\frac{3}{2}}+x^{\frac{3}{2}}\right)$.

2. Find the factors of a^2-x^2.

Ans. $(a-x)(a+x)$, or $\left(a^{\frac{1}{2}}-x^{\frac{1}{2}}\right)\left(a^{\frac{3}{2}}+ax^{\frac{1}{2}}+a^{\frac{1}{2}}x+x^{\frac{3}{2}}\right)$.

3. Find the factors of a^6-x^6.

Ans. $(a-x)(a^5+a^4x+a^3x^2+a^2x^3+ax^4+x^5)$, or $(a^3-x^3)(a^3+x^3)$.

4. Find the factors of a^7-x^7.

Ans. $(a-x)(a^6+a^5x+a^4x^2+a^3x^3+a^2x^4+ax^5+x^6.)$

5. Find the factors of a^6-x^6.

Ans. $(a^2-x^2)(a^4+a^2x^2+x^4)$, or $\left(a^{\frac{3}{2}}-x^{\frac{3}{2}}\right)\left(a^{\frac{9}{2}}+a^3x^{\frac{3}{2}}+a^{\frac{3}{2}}x^3+x^{\frac{9}{2}}\right)$

6. Find the factors of $a^{10}-x^{10}$.

Ans. $(a-x)(a^9+a^8x+a^7x^2+a^6x^3+a^5x^4+a^4x^5+a^3x^6+a^2x^7+ax^8+x^9)$.

7. Find the factors of $a^{10}-x^{10}$.

Ans. $(a^2-x^2)(a^8+a^6x^2+a^4x^4+a^2x^6+x^8)$, or $(a^5-x^5)(a^5+x^5)$.

8. Find the factors of $m-n$.

Ans. $(m^{\frac{1}{3}}-n^{\frac{1}{3}})(m^{\frac{2}{3}}+m^{\frac{1}{3}}n^{\frac{1}{3}}+n^{\frac{2}{3}})$, or $(m^{\frac{1}{2}}-n^{\frac{1}{2}})(m^{\frac{1}{2}}+n^{\frac{1}{2}})$.

9. Find the factors of x^8-y^4.

Ans. $(x^2-y)(x^6+x^4y+x^2y^2+y^3)$, or $(x^4-y^2)(x^4+y^2)$.

10. Find the factors of $x^{-4}-y^{\frac{4}{3}}$.

Ans. $(x^{-1}-y^{\frac{1}{3}})(x^{-3}+x^{-2}y^{\frac{1}{3}}+x^{-1}y^{\frac{2}{3}}+y)$, or $(x^{-2}-y^{\frac{2}{3}})\left(\frac{1}{x^2}+y^{\frac{2}{3}}\right)$.

11. Find the factors of x^m-y^n.

Ans. $\left(x^{\frac{m}{4}}-y^{\frac{n}{4}}\right)\left(x^{\frac{3m}{4}}+x^{\frac{m}{2}}y^{\frac{n}{4}}+x^{\frac{m}{4}}y^{\frac{n}{2}}+y^{\frac{3n}{4}}\right)$.

12. Find the factors of $x^{\frac{2}{3}}-y^{-\frac{3}{4}}$.

Ans. $\left(x^{\frac{2}{15}}-\frac{1}{y^{\frac{3}{20}}}\right)\left(x^{\frac{8}{15}}+\frac{x^{\frac{2}{5}}}{y^{\frac{3}{20}}}+\frac{x^{\frac{4}{15}}}{y^{\frac{3}{10}}}+\frac{x^{\frac{2}{15}}}{y^{\frac{9}{10}}}+\frac{1}{y^{\frac{3}{5}}}\right)$.

THEOREM XI.

(119.) The expression $x^{\frac{m}{n}}-y^{\frac{m}{n}}=\left(x^{\frac{m}{rn}}+y^{\frac{t}{rs}}\right)\left(x^{\frac{m}{n}-\frac{m}{rn}}-x^{\frac{m}{n}-\frac{2m}{rn}}y^{\frac{t}{rs}}+x^{\frac{m}{n}-\frac{3m}{rn}}y^{\frac{2t}{rs}}-\ldots\ldots-x^{\frac{2m}{rn}}y^{\frac{t}{s}-\frac{3t}{rs}}+x^{\frac{m}{rn}}y^{\frac{t}{s}-\frac{2t}{rs}}-y^{\frac{t}{s}-\frac{t}{rs}}\right)$ when r is an even number, which, when $\frac{m}{n}$, $\frac{t}{s}$, and r respectively equals u, becomes $x^u-y^u=(x+y)(x^{u-1}-x^{u-2}y+x^{u-3}y^2-\ldots\ldots-x^2y^{u-3}+xy^{u-2}-y^{u-1}$.

DEMONSTRATION.

The truth of this Theorem depends on the truth of Theorem V., and the fact that the divisor multiplied by the quotient equals the dividend.

PROBLEM

1. Find the factors of x^6-y^6.

SOLUTION.

In this example $u=6$, and making $r=6$, we have, by the Theorem, $x^6-y^6=(x+y)(x^5-x^4y+x^3y^2-x^2y^3+xy^4-y^5)$. Making $r=2$, we have $x^6-y^6=(x^3+y^3)(x^3-y^3)$.

Other factors may be obtained by assuming r equal to other even numbers.

PROBLEM

2. Find the factors of $x^{\frac{3}{4}} - y^{\frac{2}{3}}$.

SOLUTION.

In this example, $\frac{m}{n} = \frac{3}{4}$, and $\frac{t}{s} = \frac{2}{3}$, and we are at liberty to make r any positive even number. Let r then equal 6; whence, $\frac{m}{rn} = \frac{1}{8}$, and $\frac{t}{rs} = \frac{1}{9}$. Hence, by the Theorem, we have,

$$x^{\frac{3}{4}} - y^{\frac{2}{3}} = \left(x^{\frac{1}{8}} + y^{\frac{1}{9}}\right)\left(x^{\frac{5}{8}} - x^{\frac{1}{2}}y^{\frac{1}{9}} + x^{\frac{3}{8}}y^{\frac{2}{9}} - x^{\frac{1}{4}}y^{\frac{1}{3}} + x^{\frac{1}{8}}y^{\frac{4}{9}} - y^{\frac{5}{9}}\right).$$

Other factors may be obtained by assuming r to be equal to other even numbers.

EXAMPLES.

1. Find the factors of $a^4 - b^4$.

   Ans. $(a+b)(a^3 - a^2b + ab^2 - b^3)$, or $(a^2+b^2)(a^2-b^2)$, &c.

2. Find the factors of $a - b$.

   Ans. $(a^{\frac{1}{4}} + b^{\frac{1}{4}})(a^{\frac{3}{4}} - a^{\frac{1}{2}}b^{\frac{1}{4}} + a^{\frac{1}{4}}b^{\frac{1}{2}} - b^{\frac{3}{4}})$, &c.

3. Find the factors of $a^2 - b^2$.

   Ans. $(a+b)(a-b)$, or $\left(a^{\frac{1}{3}} + b^{\frac{1}{3}}\right)\left(a^{\frac{5}{3}} - a^{\frac{4}{3}}b^{\frac{1}{3}} + ab^{\frac{2}{3}} - a^{\frac{2}{3}}b + a^{\frac{1}{3}}b^{\frac{4}{3}} - b^{\frac{5}{3}}\right)$, &c.

4. Find the factors of $a^2 - b^6$.

   Ans. $\left(a^{\frac{1}{3}} + b\right)\left(a^{\frac{5}{3}} - a^{\frac{4}{3}}b + ab^2 - a^{\frac{2}{3}}b^3 + a^{\frac{1}{3}}b^4 - b^5\right)$, &c.

5. Find the factors of $a^{-4} - b^4$.

   Ans. $\left(\frac{1}{a} + b\right)\left(\frac{1}{a^3} - \frac{b}{a^2} + \frac{b^2}{a} - b^3\right)$, &c.

6. Find the factors of $a^{-6} - b^{-12}$.

   Ans. $\left(\frac{1}{a} + \frac{1}{b^2}\right)\left(\frac{1}{a^5} - \frac{1}{a^4b^2} + \frac{1}{a^3b^4} - \frac{1}{a^2b^6} + \frac{1}{ab^8} - \frac{1}{b^{10}}\right)$, &c.

7. Find the factors of $a^{\frac{4}{5}} + b^8$.

   Ans. $\left(a^{\frac{1}{5}} - b^2\right)\left(a^{\frac{3}{5}} - a^{\frac{2}{5}}b^2 + a^{\frac{1}{5}}b^4 - b^6\right)$, &c.

8. Find the factors of $16a^8 - b^{16}$.

   Ans. $(4a^2 + b^4)(8a^6 - 4a^4b^4 + 2a^2b^8 - b^{12})$. &c.

9. Find the factors of $a^{4m}-x^{8n}$.

Ans. $(a^m+x^{2n})(a^{3m}-a^{2m}x^{2n}+a^mx^{4n}-x^{6n})$, &c.

10. Find the factors of a^x-b^y.

Ans. $\left(a^{\frac{x}{4}}+b^{\frac{y}{4}}\right)\left(a^{\frac{3x}{4}}-a^{\frac{x}{2}}b^{\frac{y}{4}}+a^{\frac{x}{4}}b^{\frac{y}{2}}-b^{\frac{3y}{4}}\right)$, &c.

11. Find the factors of $a^{\frac{2}{3}}-b^{-\frac{1}{4}}$.

Ans. $\left(a^{\frac{1}{6}}+\frac{1}{b^{16}}\right)\left(a^{\frac{1}{2}}-\frac{a^{\frac{1}{3}}}{b^{\frac{1}{16}}}+\frac{a^{\frac{1}{6}}}{b^{\frac{1}{8}}}-\frac{1}{b^{\frac{3}{16}}}\right)$, &c.

12. Find the factors of $a^{-\frac{4m}{n}}-b^{\frac{4t}{3}}$.

Ans. $\left(\frac{1}{a^{\frac{m}{n}}}+b^{\frac{t}{s}}\right)\left(\frac{1}{a^{\frac{3m}{n}}}-\frac{b^{\frac{t}{s}}}{a^{\frac{2m}{n}}}+\frac{b^{\frac{2t}{s}}}{a^{\frac{m}{n}}}-b^{\frac{3t}{s}}\right)$, &c.

THEOREM XII.

(120.) The expression $x^{\frac{m}{n}}+y^{\frac{t}{s}}=\left(x^{\frac{m}{rn}}+y^{\frac{rt}{rs}}\right)\left(x^{\frac{m}{n}-\frac{m}{rn}}-x^{\frac{m}{n}-\frac{2m}{rn}}y^{\frac{t}{rs}}\right.$
$\left.+x^{\frac{m}{n}-\frac{3m}{rn}}y^{\frac{2t}{rs}}-\ldots\ldots\ldots\ldots\ldots+x^{\frac{2m}{rn}}y^{\frac{t}{s}-\frac{3t}{rs}}-x^{\frac{m}{n}}y^{\frac{t}{s}-\frac{2t}{rs}}+y^{\frac{t}{s}-\frac{t}{rs}}\right)$,
when r is an odd number, and, when $\frac{m}{n}$, $\frac{t}{s}$, and r respectively equals u, it becomes

$x^u+y^u=(x+y)\,(x^{u-1}-x^{u-2}y+x^{u-3}y^2-x^{u-4}y^3+\ldots\ldots\ldots+x^2y^{u-3}$
$-xy^{u-2}+y^{u-1}$.

DEMONSTRATION.

The truth of this Theorem depends on Theorem VI., and the fact that the divisor multiplied by the quotient equals the dividend.

PROBLEM

1. Find the factors of a^5+b^5.

SOLUTION.

In this example $u=5$, and making $r=5$, we have by the Theorem, $a^5+b^5=(a+b)(a^4-a^3b+a^2b^2-ab^3+b^4)$. Making $r=3$, we have $a^5+b^5=\left(a^{\frac{5}{3}}+b^{\frac{5}{3}}\right)\left(a^{\frac{10}{3}}-a^{\frac{5}{3}}b^{\frac{5}{3}}+b^{\frac{10}{3}}\right.$.

Other factors may be obtained by assuming r equal to other odd numbers.

PROBLEM

2. Find the factors of $x^{\frac{2}{3}}+y^{\frac{2}{3}}$.

SOLUTION.

In this example $\frac{m}{n}=\frac{3}{4}$, and $\frac{t}{s}=\frac{2}{3}$, and we are at liberty to make r any positive odd number. Let r then equal 5, whence $\frac{m}{rn}=\frac{3}{20}$, and $\frac{t}{rs}=\frac{2}{15}$. Hence, by the Theorem, we have $x^{\frac{3}{4}}+y^{\frac{2}{3}}=\left(x^{\frac{3}{20}}+y^{\frac{2}{15}}\right)$ $\left(x^{\frac{3}{5}}-x^{\frac{9}{20}}y^{\frac{2}{15}}+x^{\frac{3}{10}}y^{\frac{4}{15}}-x^{\frac{3}{20}}y^{\frac{2}{5}}+y^{\frac{8}{15}}\right)$.

Other factors may be obtained by assuming r to be equal to other positive odd numbers.

EXAMPLES.

1. Find the factors of a^3+x^3. *Ans.* $(a+x)(a^2-ax+x^2)$, &c.

2. Find the factors of a^3+x^3.

Ans. $\left(a^{\frac{3}{5}}+x^{\frac{3}{5}}\right)\left(a^{\frac{12}{5}}-a^{\frac{9}{5}}x^{\frac{3}{5}}+a^{\frac{6}{5}}x^{\frac{6}{5}}-a^{\frac{3}{5}}x^{\frac{9}{5}}+x^{\frac{12}{5}}\right)$, &c.

3. Find the factors of a^7+a^7.

Ans. $\left(a^{\frac{7}{3}}+x^{\frac{7}{3}}\right)\left(a^{\frac{14}{3}}-a^{\frac{7}{3}}b^{\frac{7}{3}}+b^{\frac{14}{3}}\right)$, &c.

4. Find the factors of a^7+a^7.

Ans. $\left(a^{\frac{7}{5}}+x^{\frac{7}{5}}\right)\left(a^{\frac{28}{5}}-a^{\frac{21}{5}}x^{\frac{7}{5}}+a^{\frac{14}{5}}x^{\frac{14}{5}}-a^{\frac{7}{5}}x^{\frac{21}{5}}+x^{\frac{28}{5}}\right)$, &c.

5. Find the factors of $a^{-3}+x^{-3}$.

Ans. $\left(\frac{1}{a}+\frac{1}{x}\right)\left(\frac{1}{x^2}-\frac{1}{ax}+\frac{1}{x^2}\right)$, &c.

6. Find the factors of $a^{-5}+x^{10}$.

Ans. $\left(\frac{1}{a}+x^2\right)\left(\frac{1}{a^4}-\frac{x^2}{a^3}+\frac{x^4}{a^2}-\frac{x^6}{a}+x^8\right)$, &c.

7. Find the factors of $\frac{32}{a^{10}}+\frac{1}{x^5}$.

Ans. $\left(\frac{2}{a^2}+\frac{1}{x}\right)\left(\frac{16}{a^8}-\frac{8}{a^6x}+\frac{4}{a^6x^2}-\frac{2}{a^4x^3}+\frac{1}{x^4}\right)$, &c.

8. Find the factors of $a^{\frac{2}{3}}+x^{\frac{7}{5}}$.

Ans. $\left(a^{\frac{2}{9}}+x^{\frac{7}{15}}\right)\left(a^{\frac{4}{9}}-a^{\frac{2}{9}}x^{\frac{7}{15}}+x^{\frac{14}{15}}\right)$, &c.

9. Find the factors of $a^{5m}+x^{10n}$.

Ans. $(a^m+x^{2n})(a^{4m}-a^{3m}x^{2n}+a^{2m}x^{4n}-a^mx^{6n}+x^{8n})$, &c.

10. Find the factors of $a^{\frac{m}{n}}+x^{\frac{t}{s}}$.

Ans. $\left(a^{\frac{m}{3n}}+x^{\frac{t}{3s}}\right)\left(a^{\frac{2m}{3n}}-a^{\frac{m}{3n}}x^{\frac{t}{3s}}+x^{\frac{2t}{3s}}\right)$, &c.

11. Find the factors of $a^{\frac{2}{3}}+x^{-\frac{4}{5}}$.

$$Ans.\ \left(a^{\frac{2}{15}}+\frac{1}{x^{\frac{4}{25}}}\right)\left(a^{\frac{8}{15}}-\frac{a^{\frac{2}{5}}}{x^{\frac{4}{25}}}+\frac{a^{\frac{4}{15}}}{x^{\frac{8}{25}}}-\frac{a^{\frac{2}{15}}}{x^{\frac{12}{25}}}+\frac{1}{x^{\frac{16}{25}}}\right), \text{ \&c.}$$

12. Find the factors of $a^{\frac{m}{n}}+32x^{-5}$.

$$Ans.\ \left(a^{\frac{m}{5n}}+\frac{2}{x}\right)\left(a^{\frac{4m}{5n}}-\frac{2a^{\frac{3m}{5n}}}{x}+\frac{4a^{\frac{2m}{5n}}}{x^2}-\frac{8a^{\frac{m}{5}}}{x^3}+\frac{16}{x^4}\right), \text{ \&c.}$$

THEOREM XIII.

(121.) The trinomial $x^{2m}+ax^m+b=(x^m+c)(x^m+d)$, when $c+d=a$ and $cd=b$; and $x^{2m}+ax^m-b=(x^m+c)(x^m-d)$, when $c-d=a$ and $cd=b$; and $x^{2m}-ax^m-b=(x^m+c)(x^m-d)$ when $d-c=a$ and $cd=b$; and $x^{2m}-ax^m+b=(x^m-c)(x^m-d)$ when $c+d=a$ and $cd=b$.

DEMONSTRATION.

By multiplying x^m+c by x^m+d, we have $x^{2m}+(c+d)x^m+cd$, which is equal to $x^{2m}+ax^m+b$, when $c+d=a$ and $cd=b$.

In the same way the other forms may be proved to be true.

PROBLEM.

Find the factors of x^2-x-30.

SOLUTION.

In this example $m=1$. Let us now seek two factors of 30, one *plus* and the other *minus*, which have a difference of 1, the minus factor being numerically the greater. It is apparent that these factors are -6 and $+5$; hence, according to the third form, we have $x^2-x-30=(x+5)(x-6)$.

EXAMPLES.

1. Find the factors of x^2+5x+6. *Ans.* $(x+2)(x+3)$.

2. Find the factors of $x^2+8x+15$. *Ans.* $(x+3)(x+5)$.

3. Find the factors of x^2+8x+7. *Ans.* $(x+1)(x+7)$.

4. Find the factors of $x^2+4x-32$. *Ans.* $(x+8)(x-4)$.

5. Find the factors of $x^2-9x+20$. *Ans.* $(x-4)(x-5)$.

6. Find the factors of $x^2-5x-66$. *Ans.* $(x+6)(x-11)$.

7. Find the factors of x^6-x^3-6. *Ans.* $(x^3+2)(x^3-3)$.

8. Find the factors of $x^2+x+\frac{2}{9}$. *Ans.* $(x+\frac{2}{3})(x+\frac{1}{3}.)$

9. Find the factors of $x^{2m}-x^m-72$. *Ans.* $(x^m+8)(x^m-9.)$

10. Find the factors of $x^2-\frac{3}{4}x-\frac{55}{64}$. *Ans.* $(x+\frac{5}{8})(x-\frac{11}{8})$.

11. Find the factors of $x^2+(a+b)x+ab$. *Ans.* $(x+a)(x+b)$.

12. Find the factors of $x^{\frac{r}{s}}+3x^{\frac{r}{2s}}+2$.

Ans. $(x^{\frac{r}{2s}}+1)(x^{\frac{r}{2s}}+2)$.

PROBLEM A.

(122.) To resolve a monomial into factors.

RULE.

Assume any monomial as one of the factors, and divide the given monomial by it to obtain the other factor.

PROBLEM.

Resolve a^2 into two factors.

SOLUTION.

Assume that a^3 is one of the factors, then $\dfrac{a^2}{a^3}=a^{-1}$ is the other.

EXAMPLES.

1. Resolve a^5 into factors. *Ans.* $a^2 \cdot a^3$, or $a^6 \cdot a^{-1}$, or $b \cdot \dfrac{a^5}{b}$, &c.

2. Resolve $a^{-2}b^3$ into factors. *Ans.* $\dfrac{1}{a^4} \cdot a^2b^3$, &c.

3. Resolve $6a$ into factors. *Ans.* $3a \cdot 2a^0$, &c.

4. Resolve $9b^2$ into factors. *Ans.* $3x^0 \cdot 3b^2$, &c.

5. Resolve $12ab$ into factors. *Ans.* $3ax \cdot 4bx^{-1}$, &c.

6. Resolve $20a^2x^{-2}$ into factors. *Ans.* $4a^6x^4 \cdot 5a^{-4}x^{-6}$, &c.

7. Resolve $(a+b)^3$ into factors. *Ans.* $(a+b)(a+b)^2$, &c.

8. Resolve $(a+b+c)^4$ into factors.

Ans. $(a+b+c)^2\dfrac{(a+b+c)^4}{(a+b+c)^2}$, &c.

9. Resolve $15ay$ into factors. *Ans.* $3a \cdot y^3 \cdot \dfrac{y}{y^3}$, &c.

10. Resolve $21a^{\frac{1}{3}}$ into factors. *Ans.* $3a\cdot\frac{7}{a^{\frac{2}{3}}}$, &c.

PROBLEM B.

(123.) To resolve a polynomial into factors one of which shall be a monomial.

RULE.

Assume any monomial as one of the factors and divide the given polynomial by it to obtain the other factor.

PROBLEM

1. Resolve a^2+x^2 into factors one of which shall be a monomial.

SOLUTION.

Assume a to be one of the factors, then $\frac{a^2+x^2}{a}=a+\frac{x^2}{a}$ is the other.

PROBLEM

2. Find the factors of $3ax^3+6bx^4+9cx^5$.

SOLUTION.

Assume $3x^3$ to be one of the factors, then $(3ax^3+6bx^4+9cx^5)\div 3x^3$ $=a+2bx+3cx^2$. Hence, $3ax^3+6bx^4+9cx^5=3x^3(a+2bx+3cx^2)$.

SCHOLIUM.—In order to obtain the simplest factors of a polynomial, assume as one of the factors, the greatest monomial that will divide the given polynomial and give a quotient containing neither a fraction, nor a negative exponent.

EXAMPLES.

1. Find the factors of $2a+2b$. *Ans.* $2(a+b)$.

2. Find the factors of $ax+bx+cx$. *Ans.* $(a+b+c)x$.

3. Find the factors of $6x^2+12x-18$. *Ans.* $6(x^2+2x-3)$.

4. Find the factors of $xy+y$. *Ans.* $(x+1)y$.

5. Find the factors of $acx+abx$. *Ans.* $(c+b)ax$.

6. Find the factors of $3x^2y+3xy^2$. *Ans.* $3xy(x+y)$.

7. Find the factors of $5a^3+10a^2b+5ab^2$.

Ans. $5a(a^2+2ab+b^2)$.

8. Find the factors of ax^m-bx^{m+1}.

Ans. $(ax-bx^2)x^{m-1}$, or $(a-bx)x^m$.

9. Find the factors of $14x^2y^2-21x^3y^2$. *Ans.* $7(2-3x)x^2y^2$.

10. Find the factors of $51x^3-17x^2+34x$.

Ans. $17(3x^2-x+2)x$.

11. Find the factors of $6a^{3n-1}b^rc^{r+2}-6a^{2n-4}b^3c^rd^{r-1}$.

Ans. $6(a^{n+3}b^{r-3}c^2-d^{r-1})a^{2n-4}b^3c^r$.

12. Find the factors of $2a^{n+r}b^{m-1}c-4a^rb^{2m-1}c^2d+2a^{r+1}b^mc+6a^{r-1}b^{m-1}c^n$. *Ans.* $2(a^{n+1}-2ab^mcd+a^2b+3c^{n-1})a^{r-1}b^{m-1}c$.

PROBLEM

(124.) 1. Find the factors of $2a^8x-4a^5x^4+2a^2x^7$.

SOLUTION.

By problem B, we have $2a^8x-4a^5x^4+2a^2x^7=2a^2x(a^6-2a^3x^3+x^6)$; but by Theorem V. $(a^6-2a^3x^3+x^6)=(a^3-x^3)^2$; and by Theorem X., $a^3-x^3=(a-x)(a^2+ax+x^2)$; whence, $(a^3-x^3)^2=(a-x)^2(a^2+ax+x^2)^2$. Therefore, $2a^8x-4a^5x^4+2a^2x^7=2a^2x(a-x)^2(a^2+ax+x^2)^2$ $=2a^2x(a-x)(a-x)(a^2+ax+x^2)(a^2+ax+x^2)$.

PROBLEM

2. Write the product of $a+y+x$ by $a-y+x$.

SOLUTION.

Since $a+y+x=(a+x)+y$, and $a-y+x=(a+x)-y$, we have only to find, considering $(a+x)$ as a single quantity, the product of the sum and difference of two quantities. Then by Theorem III., we have $(a+y+x)(a-y+x)=(a+x)^2-y^2$.

PROBLEM

3. Find the factors of $[(a+b)^2-c^2]^2-d^2$.

SOLUTION.

Considering what is within the brackets, as one quantity, the given expression represents the difference of two quantities, which may be factored by theorem IX., X., or XI. Let us use Theorem IX., which gives $[(a+b)^2-c^2+d][(a+b)^2-c^2-d]$. Also, by the same Theorem, we have $(a+b)^2-c^2=(a+b+c)(a+b-c)$; hence, $[(a+b)^2-c^2]^2-d^2=[(a+b+c)(a+b-c)+d][(a+b+c)(a+b-c)-d]$.

MISCELLANEOUS EXAMPLES.

1. Square $a+b+c$. *Ans.* $(a+b)^2+2(a+b)c+c^2$.

2. Square $a+b-c-d$. *Ans.* $(a+b)^2-2(a+b)(c+d)+(c+d)^2$.

3. Multiply $a+b+c$ by $b-a+c$. *Ans.* $(b+c)^2-a^2$.

4. Factor $(a+b)^3-(c-d)^3$.
Ans. $(a+b-c+d)[(a+b)^2+(a+b)(c-d)+(c-d)^2]$.

5. Factor $m^2+2mn+n^2-a^2+2ab-b^2$.
Ans. $(m+n+a-b)(m+n-a+b)$.

6. Factor $(x+y)^4-r^8$.
Ans. $(x+y+r^2)(x+y-r^2)[(x+y)^2+r^4]$.

7. Factor $a^5+(b+c)^5$.
Ans. $(a+b+c)[a^4-a^3(b+c)+a^2(b+c)^2-a(b+c)^3+(b+c)^4]$.

8. Factor $ac+ad+bd+bc$.
Ans. $a(c+d)+b(c+d)=(a+b)(c+d)$.

9. Factor $am+2bx+2ax+bm$. *Ans.* $(a+b)(m+2x)$.

10. Factor $5a^3+10a^2b+5ab^2$. *Ans.* $5a(a+b)(a+b)$.

11. Factor $3x^2+6xy+3y^2$. *Ans.* $3(x+y)(x+y)$.

12. Factor a^3-ab^2. *Ans.* $a(a+b)(a-b)$.

13. Factor $x^3-x^2y-xy^2+y^3$. *Ans.* $(x+y)(x-y)(x-y)$.

14. Factor $a^3-2a^2b+2ab^2-b^3$. *Ans.* $(a-b)(a^2-ab+b^2.)$

15. Factor $a^4+a^2b^2+b^4$. *Ans.* $(a^2+ab+b^2)(a^2-ab+b^2)$.

16. Factor $a^3-3a^2x+3ax^2-x^3$. *Ans.* $(a-x)(a-x)(a-x)$.

17. Factor $7x^2-12x+5$. *Ans.* $(x-1)(7x-5)$.

18. Factor x^3-x^2-2x. *Ans.* $x(x+1)(x-2)$.

19. Factor $2x^3+3x^2+x$. *Ans.* $x(x+1)(2x+1)$.

20. Factor $a^2-b^2-c^2-2bc$. *Ans.* $(a+b+c)(a-b-c)$.

(125.) USEFUL FORMULAS.

1. $(a+b)^2=a^2+2ab+b^2$.

2. $(a-b)^2=a^2-2ab+b^2$.

3. $(a+b)(a-b)=a^2-b^2$.

4. $(a^2+ab+b^2)(a-b)=a^3-b^3$.

5. $(a^2-ab+b^2)(a+b)=a^3+b^3.$

6. $(a^3-a^2b+ab^2-b^3)(a+b)=a^4-b^4.$

7. $a^2+2ab+b^2=(a+b)(a+b).$

8. $a^2-2ab+b^2=(a-b)(a-b).$

9. $a^2-b^2=(a+b)(a-b).$

10. $a^3-b^3=(a-b)(a^2+ab+b^2).$

11. $a^4-b^4=(a+b)(a^3-a^2b+ab^2-b^3).$

12. $a^6-b^6=(a-b)(a^5+a^4b+a^3b^2+a^2a^3+ab^4+b^5).$

13. $a^6-b^6=(a+b)(a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5).$

14. $a^6-b^6=(a^2-b^2)(a^4+a^2b^2+b^4).$

15. $a^6-b^6=(a^3+b^3)(a^3-b^3).$

16. $a^6-b^6=(a+b)(a-b)(a^2+ab+b^2)(a^2-ab+b^2).$

17. $a^4+a^2b^2+b^4=(a^2+ab+b^2)(a^2-ab+b^2).$

18. $a^3+b^3=(a+b)(a^2-ab+b^2).$

19. $a^5+a^4b+a^3b^2+a^2b^3+ab^4+b^5=(a+b)(a^2+ab+b^2)(a^2-ab+b^2)$
$=(a+b)(a^4+a^2b^2+b^4).$

20. $a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5=(a-b)(a^2+ab+b^2)(a^2-ab+b^2)$
$=(a-b)(a^4+a^2b^2+b^4).$

CHAPTER IV.

GREATEST COMMON DIVISOR.

(126.) A *multiple* of a quantity, is a quantity that contains it an an exact number of times. Thus, 6 is a multiple of 2 and 3, and ab is a multiple of a and b.

(127.) A *measure* of a quantity, is a quantity that is contained in it an exact number of times. Thus, 2 and 3 are measures of 6, and a and b are measures of ab.

(128.) A *common measure* or *common divisor* of two or more quantities is one that is exactly contained in each of them.

(129.) The *greatest common measure* or *greatest common divisor* of two or more quantities is the greatest quantity that is exactly contained in each of them.

THEOREM I.

(130.) *If a quantity measures another quantity it will measure any multiple of that quantity.*

DEMONSTRATION.

Let a be a quantity, then ta is a multiple of a.

We are to prove that if a quantity d measures a it will also measure ta. Let r be the number of times d is contained in a; whence, we have, $a=rd$ and $ta=trd$. Now d measures trd and must, therefore, measure ta, which is equal to trd.

THEOREM II.

(131.) *If a quantity measures two other quantities, it will also measure their sum and their difference.*

DEMONSTRATION.

Let Rd and rd be two quantities that are divisible by d.

We are to prove that $Rd+rd$ and $Rd-rd$ are both divisible by d. Since $Rd+rd=d(R+r)$ and $Rd-rd=d(R-r)$, we see that d is a factor in both these expressions, and is, therefore, a divisor of both.

PROBLEM.

(132.) To find the greatest common divisor of two or more monomials.

RULE.

Resolve the monomials into their factors, and the product of the factors common to all the monomials will be the greatest common divisor.

PROBLEM.

(133.) Find the greatest common divisor of $3a^2b^3d^4$, $6ab^2d^3$, and $12bd^2$.

SOLUTION.

Resolving these monomials into factors, we have

$$\begin{aligned} 3a^2b^2d^4 &= a^2bd^2 \cdot 3bd^2, \\ 6\,ab^2d^3 &= 2\,abd \cdot 3bd^2, \\ \text{and} \quad 12bd^2 &= 4 \cdot 3bd^2. \end{aligned}$$

From which it is seen that $3bd^2$ contains all the common factors, and is, therefore, the greatest common divisor.

EXAMPLES.

1. Find the greatest common divisor of $12ab^3c$ and $25b^4c^5$.

Ans. b^3c.

2. Find the greatest common divisor of $3a^2b^2c^2$ and $6a^4b^4c^3$.

Ans. $3a^2b^2c^2$.

3. Find the greatest common divisor of $3a^2b^{\frac{2}{3}}$ and $2abcd^{\frac{4}{5}}$.

Ans. $ab^{\frac{2}{3}}$.

4. Find the greatest common divisor of $49a^2b^2c^4$ and $63a^5b^3c^3$.

Ans. $7a^2b^2c^3$.

5. Find the greatest common divisor of $a^{\frac{1}{2}}$ and $a^{\frac{1}{3}}$.

6. Find the greatest common divisor of $a^{-\frac{1}{2}}$ and $a^{\frac{1}{3}}$.

7. Find the greatest common divisor of $a^{-\frac{1}{2}}$ and $-\frac{1}{3}$.

REMARK.—The last three examples are left unanswered to call out thought.

PROBLEM.

(134.) To find the greatest common divisor of two polynomials.

RULE.

1. *Find the greatest monomial factor that is contained in both polynomials, and reserve it.*

2. *Reject the remaining monomial factors from each polynomial.*

3. *Arrange the terms of the resulting polynomials according to the powers of some letter in both, and consider that polynomial of which the leading letter in the first term has the least exponent as the divisor, and the other polynomial as the dividend.*

4. *Multiply the dividend by the least monomial that will render the first term of the dividend exactly divisible by the first term of the divisor.*

5. *Divide the dividend by the divisor, and continue the division until the highest exponent of the leading letter in the remainder is less than the highest exponent of the leading letter in the divisor.* [*If the coefficient of the first term in any remainder is not divisible by the coefficient of the first term in the divisor, to avoid fractions multiply the remainder by such a number as will render its first coefficient exactly divisible by the coefficient of the first term of the divisor.*]

6. *Reject from the remainder its greatest monomial factor, and then consider the result a new divisor, and the former divisor a new dividend, proceed as before, and continue the process until the remainder is zero.*

7. *Multiply the last divisor by the reserved monomial, and the product will be the greatest common divisor of the given polynomials.*

DEMONSTRATION.

Let A and B represent two polynomials of which we seek the greatest common divisor. Let C and D represent two other polynomials, neither of which can be divided by a monomial.

Suppose $A=a^3bC$ and $B=a^2cD$; a, b, and c being monomials.

1. It is evident that the greatest monomial factor common to a^3bC and a^2cD is a^2, which must be reserved, because it evidently is a factor of the greatest common divisor of A and B, or, which is the same, of a^3bC and a^2cD.

2. We have left abC and cD. We now seek the greatest common divisor of these polynomials. Since a is not a factor common to both these polynomials, it can not be a factor of their *common* divisor, and therefore may be rejected. For the same reason b and c may be rejected. Hence the greatest common divisor of abC and cD is the same as the greatest common divisor of C and D.

3. Suppose that the terms of the polynomials C and D are arranged according to the powers of the same letter in both, and that the exponent of the leading letter in the first term of C is less than the exponent of the same letter in the first term of D. Therefore, consider C as a divisor and D as a dividend.

4 Suppose the first term of C contains the monomial $3m$ and that the first term of D does not. Then, in order that the first term of the quotient should not be fractional, the polynomial D should be multiplied by $3m$. The greatest common divisor of C and D is the same as the greatest common divisor of C and $3mD$, since the introduced monomial $3m$ can form no part of the greatest common divisor of C and $3mD$, because by hypothesis $3m$ can not be a factor of C.

Operation.

$$\begin{array}{l}
C)3mD(q \\
\quad \underline{qC} \\
\quad\;\; E \\
\quad\;\; 2 \\
C)\overline{2E}(q' \\
\quad \underline{q'C} \\
5p)\;F \\
\quad \overline{\;G)2nC(q''} \\
\qquad\quad \underline{q''G} \\
\qquad\qquad 0
\end{array}$$

5. Divide $3mD$ by C, and suppose the first term of the quotient to be q, and the first remainder E. Again, suppose that the first coefficient of C contains the factor 2, and that the first coefficient of E does not. Then multiply E by 2 and divide the result $2E$ by C and let q' represent the first term of the quotient, and F the remainder. Also suppose that the exponent of the leading letter in the first term of F is less than the exponent of the same letter in the first term of C.

6. Suppose that the greatest monomial factor of F is $5p$. Reject this factor, or, in other words, divide F by $5p$, and let G represent the result, which consider as a new divisor and C as a new dividend. Suppose, then, the first term of G contains the monomial $2n$ and the first term of C does not. Then, in order that the first term of the quotient should not be fractional, the polynomial C should be multiplied by $2n$.

Let q'' be the exact quotient of $2nC$ by G.

7. G is the greatest common divisor of C and D, and multiplying

it by the reserved monomial a^2, we have a^2G for the greatest common divisor of A and B.

We prove that G is the greatest common divisor of C and D as follows :—

Let G' be the greatest common divisor of C and D. We have already shown that the greatest common divisor of C and $3nD$ is the same as the greatest common divisor of C and D.

Since G' is a measure of C and D, it must (**130**) be a measure of qC and $3mD$, and it must also (**131**) be a measure of E, the difference between qC and $3mD$.

Since G' is a measure of C and E, it must also (**130**) be a measure of $q'C$ and $2E$, and it must also (**131**) be a measure of F, the difference between $q'C$ and $2E$.

Because, by hypothesis, C and D contain no monomial factors, it follows that G', their greatest common divisor, is neither a monomial nor divisible by a monomial.

But G' measures F, and consequently must measure G, which represents F after its monomial factors are rejected. Hence, the greatest common measure of C and D can not be greater than G, and, therefore, if G is a common measure of C and D, it must be the greatest.

Since G measures $2nC$, and is not a monomial, it must also measure C, and consequently must measure $q'C$.

But G also measures F, therefore it must measure $2E$, which is the sum of F and $q'C$. Because, G measures $2E$, and is not a monomial, it must also measure E: G must also measure qC, and therefore must measure $3mD$, which is the sum of E and qC. Since G measures $3mD$, and is not a monomial, it must also measure D. Hence, G is a common measure of C and D, and must, therefore, as shown above, be the greatest.

Now let $C=QG$ and $D=Q'G$, and we have

$A=a^3bQG$ and $B=a^2cQ'G$; whence, we see that a^2G must be the common measure of A and B, because Q and Q' may be rejected, since they can have no common measure.

Therefore the truth of the rule is established.

PROBLEM.

(**135.**) Find the greatest common divisor of the polynomials, $6a^3b-6a^2by-2by^3+2aby^2(A)$ and $12a^2b+3by^2-15aby$ (B).

SOLUTION.

1. We see by inspection that b is the greatest monomial that is common to both polynomials, which reserve.

2. By rejecting the monomial 2 from (A), and 3 from (B), we have $3a^3-3a^2y-y^3+ay^2$ and $4a^2+y^2-5ay$.

3. Arranging these results according to the powers of the letter a, we have $3a^3-3a^2y+ay^2-y^3$ and $4a^2-5ay+y^2$. Hence, the latter must be the divisor.

Operation.

$$
\begin{array}{l}
\qquad\qquad\quad 3a^3-\ 3a^2y+\ \ ay^2-\ \ y^3 \\
\qquad\qquad\quad \underline{4\qquad\qquad\qquad\qquad\quad} \\
4a^2-5ay+y^2)12a^3-12a^2y+\ 4ay^2-\ 4y^3(3a \\
\qquad\qquad\quad \underline{12a^3-15a^2y+\ 3ay^2} \\
\qquad\qquad\qquad\qquad 3a^2y+\ \ ay^2-\ 4y^3 \\
\qquad\qquad\qquad\qquad \underline{4\qquad\qquad\qquad\quad} \\
\qquad\qquad\qquad\quad 12a^2y+\ 4ay^2-16y^3(3y \\
\qquad\qquad\qquad\quad \underline{12a^2y-15ay^2+\ 3y^3} \\
\qquad\qquad\qquad\qquad 19y^2)\underline{19ay^2-19y^3} \\
\qquad\qquad\qquad\qquad\qquad\quad a-y)4a^2-5ay+y^2(4a-y \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{4a^2-4ay} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad -\ ay+y^2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{-\ ay+y^2} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0
\end{array}
$$

4. Since the first term of the divisor is not contained in the dividend, we multiply the dividend by 4, and then dividing, we have for the first remainder $3a^2y+ay^2-4y^3$, which must be multiplied by 4 to avoid a fractional result in the quotient. Continuing the division, we have for the next remainder $19ay^2-19y^3$, in which the exponent of the leading letter a is less than the exponent of the same letter in the first term of the divisor.

5. Rejecting from this remainder $19y^2$, the greatest monomial contained in it, we have $a-y$, which constitutes the new divisor, which divided into $4a^2-5ay+y^2$, leaves zero for a remainder.

7. Hence, the last divisor $a-y$ multiplied by the reserved monomial b, gives $b(a-y)$ for the greatest common divisor of the given polynomials (A) and (B).

EXAMPLES.

1. Find the greatest common divisor of $x^4-8x^3+21x^2-20x+4$ and $2x^3-12x^2+21x-10$. *Ans.* $x-2$.

2. Find the greatest common divisor of $3a^2-2a-1$ and $4a^3-2a^2-3a+1$. *Ans.* $a-1$.

3. Find the greatest common divisor of a^3-b^3 and $a^3+2a^2b+2ab^2+b^3$. *Ans.* a^2+ab+b^2.

4. Find the greatest common divisor of $a^4+a^3b-ab^3-b^4$ and $a^4+a^2b^2+b^4$. *Ans.* a^2+ab+b^2.

5. Find the greatest common divisor of $a^2-2ax+x^2$ and $a^3-a^2x-ax^2+x^3$. *Ans.* $a^2-2ax+x^2$.

6. Find the greatest common divisor of $x^3+9x^2+27x-98$ and $x^2+12x-28$. *Ans.* $x-2$.

7. Find the greatest common measure of $7a^2-23ab+6b^2$ and $5a^3-18a^2b+11ab^2-6b^3$. *Ans.* $a-3b$.

8. Find the greatest common divisor of $6a^5+15a^4b-4a^3c^2-10a^2bc^2$ and $9a^3b-27a^2bc-6abc^2+18bc^3$. *Ans.* $3a^2-2c^2$.

9. Find the greatest common divisor of $36a^6b^2-18a^5b^2-27a^4b^2+9a^3b^2$ and $27a^5b^2-18a^4b^2-9a^3b^2$. *Ans.* $9a^3b^2(a-1)$.

10. Find the greatest common divisor of $(c-d)a^2+(2bc-2bd)a+(b^2c-b^2d)$ and $(bc-bd+c^2-cd)a+(b^2d+bc^2-b^2c-bcd)$.
Ans. $c-d$.

PROBLEM.

(136.) To find the greatest common divisor of two or more polynomials.

RULE.

Resolve all the polynomials into their simplest factors, and take the product of those which are common, for the greatest common divisor.

DEMONSTRATION.

The truth of this rule is a consequence of the definition of the greatest common divisor.

PROBLEM.

(137.) Find the greatest common divisor of $x^2-bx+2ax-2ab$ and $x^3+ax^2+bx^2-2a^2x+bax-2ba^2$.

SOLUTION.

Arranging $x^2-bx+2ax-2ab$, we have $x^2+2ax-bx-2ab$.

Factoring, we obtain $(x+2a)x-b(x+2a)=(x+2a)(x-b)$.

Arranging, $x^3+ax^2+bx^2-2a^2x+bax-2ba^2$, we have $x^3+ax^2-2a^2x+bx^2+bax-2ba^2$.

Since, $ax^2=2ax^2-ax^2$ and $bax=2bax-bax$, we have $x^3+ax^2-2a^2x+bx^2+bax-2ba^2=x^3+2ax^2-ax^2-2a^2x+bx^2+2bax-bax-2ba^2$.

Factoring, we obtain $(x+2a)x^2-ax(x+2a)+bx(x+2a)-ab(x+2a)=(x+2a)(x^2-ax+bx-ab)$.

But, $x^2-ax+bx-ab=(x-a)x+b(x-a)=(x-a)(x+b)$.

Hence, we have $x^3+ax^2+bx^2-2a^2x+bax-2ba^2=(x+2a)(x-a)(x+b)$.

We also have $x^2-bx+2ax-2ab=(x+2a)(x-b)$.

Whence we see that $x+2a$ is the only common factor.

EXAMPLES.

1. Find the greatest common divisor of x^3-a^3, x^4-a^4, and x^2-a^2.
Ans. $x-a$.

2. Find the greatest common divisor of a^4-x^4, $a^3-a^2x-ax^2+x^3$, and $2a^3b-2abx^2$.
Ans. a^2-x^2.

3. Find the greatest common divisor of $5a^5+10a^4b+5a^3b^2$, $a^3b+2a^2b^2+2ab^3+b^4$, and a^2-b^2.
Ans. $a+b$.

4. Find the greatest common divisor of $x^4+ax^3-a^3x-a^4$ and $x^4+a^2x^2+a^4$.
Ans. x^2+ax+a^2.

5. Find the greatest common divisor of x^4-1, x^5+x^3, x^8-1, and x^4+2x^2+1.
Ans. x^2+1.

6. Find the greatest common divisor of x^3-b^2x and $x^2+2bx+b^2$.
Ans. $x+b$.

7. Find the greatest common divisor of $3a^3-3a^2b+ab^2-b^3$ and $4a^2b-5ab^2+b^3$.
Ans. $a-b$.

8. Find the greatest common divisor of $a^2-5ab+4b^2$, $a^3-a^2b+3ab^2-3b^3$, and a^4-b^4.
Ans. $a-b$.

9. Find the greatest common divisor of $12x^4-24x^3y+12x^2y^2$ and $8x^3y^2-24x^2y^3+24xy^4-8y^5$.
Ans. $4(x^2-2xy+y^2)$.

10. Find the greatest common divisor of $2x^2+3ax+a^2$, $2ax^2-a^2x-a^3$, and $6x^2+3ax$.
Ans. $2x+a$.

CHAPTER V.

LEAST COMMON MULTIPLE.

(138.) The LEAST COMMON MULTIPLE of two or more quantities is the least quantity that can be measured by all of them.

PROBLEM.

(139.) To find the least common multiple of two or more quantities.

RULE.

Resolve the given quantities into their simplest factors, and find the continued product of all the different factors, taking each factor the greatest number of times that it occurs as a factor in any of the given quantities, and the result will be the least common multiple sought.

DEMONSTRATION.

The least common multiple must contain as many factors as are contained in any of the given quantities.

PROBLEM.

(140.) Find the least common multiple of a^2-x^2, $a^2-2ax+x^2$, and a^4-x^4.

SOLUTION.

Factoring these quantities, we have
$(a+x)(a-x)$, $(a-x)(a-x)$, and $(a-x)(a+x)(a^2+x^2)$.

Hence, the least common multiple is $(a+x)(a-x)(a-x)(a^2+x^2)$ $=a^5-ax^4-a^4x+x^5$.

EXAMPLES.

1. Find the least common multiple of $2a^2x$ and $10a^3x^3$.

Ans. $10a^3x^3$.

2. Find the least common multiple of $6ax^2$, $4a^2b^3x$, and $8a^3b^2xy$.

Ans. $24a^3b^3x^2y$.

3. Find the least common multiple of x^2-y^2 and x^3-y^3.
Ans. $(x+y)(x^3-y^3)$.

4. Find the least common multiple of a^2-b^2 and a^3+b^3.
Ans. $(a-b)(a^3+b^3)$.

5. Find the least common multiple of x^2-8x+7 and x^2+7x-8.
Ans. $x^3-57x+56$.

6. Find the least common multiple of $x^3-x^2y-xy^2+y^3$, $x^3-x^2y+xy^2-y^3$ and x^4-y^4. *Ans.* $x^5-xy^4-x^4y+y^5$.

7. Find the least common multiple of $(a+b)^2$, a^2-b^2, $(a-b)^2$, and $a^3+3a^2b+3ab^2+b^3$. *Ans.* $(a+b)(a^2-b^2)^2$.

8. Find the least common multiple of $a+b$, $a-b$, a^2+ab+b^2, and a^2-ab+b^2. *Ans.* a^6-b^6.

9. Find the least common multiple of $a^3+3a^2b+3ab^2+b^3$, $a^2+2ab+b^2$, and a^2-b^2. *Ans.* $a^4+2a^3b-2ab^3-b^4$.

10. Find the least common multiple of $x^4-5x^3+9x^2-7x+2$, and x^4-6x^2+8x-3. *Ans.* $x^5-2x^4-6x^3+20x^2-19x+6$.

CHAPTER VI.

ALGEBRAIC FRACTIONS.

(141.) AN ALGEBRAIC FRACTION is an algebraic expression denoting the division of one quantity by another, and is written in the same manner as a common fraction in arithmetic.

An algebraic fraction may also be considered as a certain part of unity.

THEOREM I.

(142.) The value of a fraction is not changed when both of its terms are multiplied by the same quantity.

DEMONSTRATION.

Let $\frac{A}{B}$ represent any algebraic fraction.

We are to prove that $\frac{A}{B}=\frac{mA}{mB}$.

By **(101)**, we have $\frac{mA}{mB}=\frac{m^1m^{-1}A}{B}=\frac{m^0A}{B}=\frac{A}{B}$, because $m^0=1$.

THEOREM II.

(143.) The value of a fraction is not changed when both of its terms are divided by the same quantity.

DEMONSTRATION.

Let $\frac{A}{B}$ represent any algebraic fraction.

We are to prove that $\frac{A}{B}=\frac{A\div m}{B\div m}$.

Since, $A\div m=\frac{A}{m}$ and $B\div m=\frac{B}{m}$, we have, by **(101)** $\frac{m^{-1}A}{m^{-1}B}=\frac{m^{-1}m^1A}{B}=\frac{m^0A}{B}=\frac{A}{B}$. *Q. E. D.*

PROBLEM.

(144.) To reduce a fraction to its lowest terms.

RULE.

Resolve both terms into their simplest factors, and cancel those that are common.

Or, *divide both terms by their greatest common divisor.*

DEMONSTRATION.

The accuracy of this rule depends upon **(143)**.

PROBLEM.

(145.) Reduce $\frac{x^4+a^2x^2+a^4}{x^4+ax^3-a^3x-a^4}$ to its lowest terms.

SOLUTION.

Factoring, we have

$$\frac{x^4+a^2x^2+a^4}{x^4+ax^3-a^3x-a^4}=\frac{(x^2+a^2+ax)(x^2+a^2-ax)}{x^4-a^4+ax^3-a^3x}=$$

$$\frac{(x^2+a^2+ax)(x^2+a^2-ax)}{(x^2+a^2)(x^2-a^2)+ax(x^2-a^2)}=\frac{(x^2+a^2+ax)(x^2+a^2-ax)}{(x^2+a^2+ax)(x^2-a^2)}.$$

Canceling the common factor, there results

$$\frac{x^2+a^2-ax}{x^2-a^2}=\frac{x^2-ax+a^2}{x^2-a^2}$$

EXAMPLES.

1. Reduce $\frac{a^2bc}{5a^2b^2}$ to its lowest terms. *Ans.* $\frac{c}{5b}$.

2. Reduce $\frac{ax}{ax+x^2}$ to its lowest terms. *Ans.* $\frac{a}{a+x}$.

3. Reduce $\frac{14x^2y^2-21x^3y^2}{7x^3y}$ to its lowest terms. *Ans.* $\frac{2y-3xy}{x}$.

4. Reduce $\frac{a+b}{a^3+b^3}$ to its lowest terms. *Ans.* $\frac{1}{a^2-ab+b^2}$.

5. Reduce $\frac{x^4-a^4}{x^5-a^2x^3}$ to its lowest terms. *Ans.* $\frac{x^2+a^2}{x^3}$.

6. Reduce $\dfrac{2x^3-16x-6}{3x^3-24x-9}$ to its lowest terms. *Ans.* $\dfrac{2}{3}$.

7. Reduce $\dfrac{a^2-2ax+x^2}{a^3-a^2x-ax^2+x^3}$ to its lowest terms. *Ans.* $\dfrac{1}{a+x}$.

8. Reduce $\dfrac{a^3-b^2a}{a^2+2ab+b^2}$ to its lowest terms. *Ans.* $\dfrac{a^2-ab}{a+b}$.

9. Reduce $\dfrac{a^3+2a^2b+3a^2b^2}{2a^4-3a^3b-5a^2b^2}$ to its lowest terms.

Ans. $\dfrac{a+2b+3b^2}{2a^2-3ab-5b^2}$.

10. Reduce $\dfrac{a^3-2a^2b+2ab^2-b^3}{a^4+a^2b^2+b^4}$ to its lowest terms.

Ans. $\dfrac{a-b}{a^2+ab+b^2}$

11. Reduce $\dfrac{y^4-x^4}{y^3-y^2x-yx^2+x^3}$ to its lowest terms. *Ans.* $\dfrac{y^2+x^2}{y-x}$.

12. Reduce $\dfrac{x^2+2x-3}{x^2+5x+6}$ to its lowest terms. *Ans.* $\dfrac{x-1}{x+2}$.

13. Reduce $\dfrac{2x^3+3x^2+x}{x^3-x^2-2x}$ to its lowest terms. *Ans.* $\dfrac{2x+1}{x+2}$.

14. Reduce $\dfrac{ac+bd+ad+bc}{af+2bx+2ax+bf}$ to its lowest terms.

Ans. $\dfrac{c+d}{f+2x}$.

15. Reduce $\dfrac{a^2+b^2+c^2+2ab+2ac+2bc}{a^2-b^2-c^2-2bc.}$ to its lowest terms.

Ans. $\dfrac{a+b+c}{a-b-c}$.

16. Reduce $\dfrac{a^2-3ab+ac+2b^2-2bc}{a^2-b^2+2bc-c^2}$ to its lowest terms.

Ans. $\dfrac{a-2b}{a+b-c}$.

17. Reduce $\dfrac{(a+b)(a+b+c)(a+b-c)}{2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4}$ to its lowest terms.

Ans. $\dfrac{a+b}{(c+a-b)(b-a+c)}$.

18. Reduce $\dfrac{x^2+(a+b)x+ab}{x^2+(b+c)x+bc}$ to its lowest terms. *Ans.* $\dfrac{x+a}{x+c}$.

19. Reduce $\dfrac{2x^3+x^2-8x+5}{7x^2-12x+5}$ to its lowest terms.

Ans. $\dfrac{2x^2+3x-5}{7x-5}$.

PROBLEM.

(146.) To reduce a fraction to an entire or mixed quantity.

RULE.

Divide the numerator by the denominator, and when there is a remainder place it over the denominator for the fractional part, and connect it with the entire part of the quotient by the sign of addition, or change all the signs of the numerator of the fractional part, and connect it with the entire part of the quotient by the sign of subtraction.

DEMONSTRATION.

The accuracy of this rule is an obvious consequence of the accuracy of the process of division.

PROBLEM.

(147.) Reduce $\dfrac{x^2y^2-y^2+xy-y-x+1}{x^2-1}$ to a mixed quantity.

SOLUTION.

Dividing the numerator by x^2-1, we get y^2 and the fractional part $\dfrac{xy-y-x+1}{x^2-1}=\dfrac{(x-1)y-1(x-1)}{x^2-1}=\dfrac{(x-1)(y-1)}{(x-1)(x+1)}=\dfrac{y-1}{x+1}$. This connected with y^2 by the sign of addition gives $y^2+\dfrac{y-1}{x+1}$, and with the sign of subtraction gives $y^2-\dfrac{-y+1}{x+1}$, or $y^2-\dfrac{1-y}{1+x}$.

EXAMPLES.

1. Reduce $\dfrac{ab+x}{b}$ to a mixed quantity. *Ans.* $a+\dfrac{x}{b}$.

2. Reduce $\dfrac{a^2+bx}{a}$ to a mixed quantity. *Ans.* $a+\dfrac{bx}{a}$.

3. Reduce $\dfrac{5ax+ab+x}{x}$ to a mixed quantity. *Ans.* $5a+1+\dfrac{ab}{x}$.

4. Reduce $\dfrac{4a+2ax+b}{a}$ to a mixed quantity. *Ans.* $4+2x+\dfrac{b}{a}$.

5. Reduce $\dfrac{x^2y-xy-x+1}{xy-y}$ to a mixed quantity. *Ans.* $x-\dfrac{1}{y}$.

6. Reduce $\dfrac{1-2a^2}{1+a}$ to a mixed quantity. *Ans.* $1-a-\dfrac{a^2}{1+a}$.

7. Reduce $\dfrac{16x+7}{5}$ to a mixed quantity. *Ans.* $3x+1+\dfrac{x+2}{5}$.

8. Reduce $\dfrac{15x^2-4x+6}{5x}$ to a mixed quantity.

Ans. $3x-\dfrac{4x-6}{5x}$.

9. Reduce $\dfrac{3a^2-9ac+b-6a}{3a}$ to a mixed quantity.

Ans. $a-3c-2+\dfrac{b}{3a}$.

10. Reduce $\dfrac{5x^3-5y^3}{x-y}$ to an entire quantity. *Ans.* $5(x^2+xy+y^2)$.

PROBLEM.

(148.) To reduce a mixed quantity to a fractional form.

RULE.

Multiply the entire part by the denominator of the fractional part and add the numerator to the product when the sign of the fraction is plus, and subtract it from it when the sign of the fraction is minus; this result placed over the denominator will give the required form.

DEMONSTRATION.

This rule indicates a process which is just the reverse of that in the last rule, and must, therefore, be accurate, because the problem to be solved is just the reverse.

PROBLEM.

(149.) Reduce $2a-\dfrac{3x-b}{c}$ to a fractional form.

SOLUTION.

Multiplying $2a$ by c gives $2ac$, and from this must be *subtracted* $3x-b$, because the sign of the fraction is minus, whence we have $\frac{2a-(3x-b)}{c}=\frac{2a-3x+b}{c}$ for the required form.

SCHOLIUM.—It should be remembered that the $3x$ in this example is *plus*, and that the *minus* sign before the fraction belongs to the whole fraction, or what is the same thing, belongs to the whole numerator as indicated in the solution by the parentheses.

EXAMPLES.

1. Reduce $1+\frac{a^2-x^2}{a^2+x^2}$ to a fractional form. *Ans.* $\frac{2a^2}{a^2+x^2}$.

2. Reduce $x+\frac{x^2}{y}$ to a fractional form. *Ans.* $\frac{x(x+y)}{y}$.

3. Reduce $4x-\frac{2x-5}{3}$ to a fractional form. *Ans.* $\frac{5(2x+1)}{3}$.

4. Reduce $3x-9-\frac{3x^2-30}{x+3}$ to a fractional form. *Ans.* $\frac{3}{x+3}$

5. Reduce $1+\frac{b^2+c^2-a^2}{2bc}$ to a fractional form.

Ans. $\frac{(b+c+a)(b+c-a)}{2bc}$.

6. Reduce $1-\frac{(a-b)^2}{a^2+b^2}$ to a fractional form. *Ans.* $\frac{2ab}{a^2+b^2}$.

7. Reduce $1-\frac{b^2+c^2-a^2}{2bc}$ to a fractional form.

Ans. $\frac{(a+b-c)(a-b+c)}{2bc}$.

8. Reduce $a-x-\frac{a^2+x^2}{a+x}$ to a fractional form. *Ans.* $-\frac{2x^2}{a+x}$.

9. Reduce $2a-x+\frac{(a-x)^2}{x}$ to a fractional form. *Ans.* $\frac{a^2}{x}$.

10. Reduce $a^2-ax+x^2-\frac{x^3}{a+x}$ to a fractional form.

Ans. $\frac{a^3}{a+x}$.

PROBLEM.

(150.) To reduce fractions to equivalent fractions having a common denominator.

RULE

1. *Multiply both terms of each fraction by the product of all the denominators except its own.*

RULE

2. *Find a common multiple (generally the least) of all the denominators, and divide it by each denominator, and then multiply both terms of each fraction by these results respectively.*

DEMONSTRATION.

The accuracy of these rules depends upon the fact that the value of a fraction is not changed when both of its terms are multiplied by the same quantity.

PROBLEM.

(151.) Reduce $\frac{1+x}{1-x}$, $\frac{1+x^2}{1-x^2}$, and $\frac{1+x^3}{1-x^3}$ to equivalent fractions having a common denominator.

SOLUTION.

Applying Rule 1, we have

$$\frac{(1+x)(1-x^2)(1-x^3)}{(1-x)(1-x^2)(1-x^3)}, \frac{(1+x^2)(1-x)(1-x^3)}{(1-x^2)(1-x)(1-x^3)}, \text{and} \frac{(1+x^3)(1-x)(1-x^2)}{(1-x^3)(1-x)(1-x^2)}.$$

Applying Rule 2, using the least common multiple of the denominators, which is, $(1+x)(1+x^3)$, or $(1+x)(1+x)(1+x+x^2)$, we have

$$\frac{(1+x)(1+x)(1+x+x^2)}{(1-x)(1+x)(1+x+x^2)}, \frac{(1-x^2)(1+x-x^2)}{(1-x^2)(1+x+x^2)}, \text{ and } \frac{(1+x^3)(1+x)}{(1-x^3)(1+x)}, \text{ or}$$

by putting the denominators in the same form, we obtain

$$\frac{(1+x)^2(1+x+x^2)}{(1+x)(1-x^3)}, \frac{(1+x^2)(1+x+x^2)}{(1+x)(1-x^3)}, \text{ and } \frac{(1+x)(1+x^3)}{(1+x)(1-x^3)}.$$

EXAMPLES.

1. Reduce $\frac{a}{b}$ and $\frac{c}{d}$ to equivalent fractions having a common denominator. *Ans.* $\frac{ad}{bd}$ and $\frac{bc}{bd}$.

2. Reduce $\frac{x}{x+y}$ and $\frac{y}{x-y}$ to a common denominator.

Ans. $\frac{x^2-xy}{x^2-y^2}$ and $\frac{xy+y^2}{x^2-y^2}$.

3. Reduce $\frac{a-b}{a+b}$, $\frac{a+b}{a-b}$, and $\frac{a^2b^2}{a^2-b^2}$ to a common denominator.

Ans. $\frac{(a-b)^2}{a^2-b^2}$, $\frac{(a+b)^2}{a^2-b^2}$, and $\frac{a^2b^2}{a^2-b^2}$.

4. Reduce $\frac{3x}{a+x}$, $\frac{a-x}{3}$, and $\frac{1}{2x}$ to a common denominator.

Ans. $\frac{18x^2}{6ax+6x^2}$, $\frac{2a^2x-2x^3}{6ax+6x^2}$, and $\frac{3a+3x}{6ax+6x^2}$.

5. Reduce $\frac{a}{b}$, $\frac{2c^2}{d}$, and $x+\frac{3a-x^2}{x}$ to a common denominator.

Ans. $\frac{adx}{bdx}$, $\frac{2bc^2x}{bdx}$, and $\frac{3abd}{bdx}$.

6. Reduce $\frac{x}{3}$, $\frac{x+1}{4}$, and $\frac{x-1}{1+x}$ to a common denominator.

Ans. $\frac{4x^2+4x}{12x+12}$, $\frac{3x^2+6x+3}{12x+12}$, and $\frac{12x-12}{12x+12}$.

7. Reduce $\frac{2x+3}{x}$ and $\frac{5x+2}{3ab}$ to a common denominator.

Ans. $\frac{6abx+9ab}{3abx}$ and $\frac{5x^2+2x}{3abx}$.

8. Reduce $\frac{a}{a^2-x^2}$, $\frac{3b}{4a-4x}$, and $\frac{5x}{a+x}$ to a common denominator.

Ans. $\frac{4a}{4a^2-4x^2}$, $\frac{3ab+3bx}{4a^2-4x^2}$, and $\frac{20ax-20x^2}{4a^2-4x^2}$.

9. Reduce $\frac{x^2}{5y}$, $a-\frac{x^2-5}{3x}$, and $7+\frac{4a-15}{2}$ to a common denominator.

Ans. $\frac{6x^3}{30xy}$, $\frac{30axy-10x^2y+50y}{30xy}$, and $\frac{60axy-15xy}{30xy}$.

10. Reduce $\frac{1}{a^2+2ax+x^2}$, $\frac{1}{a^2-x^2}$, and $\frac{5y}{a^4-x^4}$ to a common denomnator.

Ans. $\frac{a^3+ax^2-a^2x-x^3}{a^5-ax^4+a^4x-x^5}$, $\frac{a^3+ax^2+a^2x+x^3}{a^5-ax^4+a^4x-x^3}$, and $\frac{5ay+5xy}{a^5-ax^4+a^4x-x^5}$.

PROBLEM.

(152.) To reduce an entire quantity to a fractional form having a given denominator.

RULE.

Multiply the entire quantity by the given denominator, and place it over the denominator.

PROBLEM.

(153.) Reduce $1+x$ to a fraction with $1-x$ for its denominator.

SOLUTION.

Multiplying $1+x$ by $1-x$, and placing the product over $1-x$, we have $\frac{1-x^2}{1-x}$ for the required form.

EXAMPLES.

1. Reduce 2 to a fraction, having 4 for its denominator. *Ans.* $\frac{8}{4}$.

2. Reduce a to a fraction, having b for its denominator. *Ans.* $\frac{ab}{b}$.

3. Reduce $2xy^4$ to a fraction, having x^3 for its denominator. *Ans.* $\frac{2x^4y^4}{x^3}$.

4. Reduce a^2+ab+b^2 to a fraction, having $a-b$ for its denominator. *Ans.* $\frac{a^3-b^3}{a-b}$.

5. Reduce $a^4+a^2x^2+x^4$ to a fraction, having a^2-x^2 for its denominator. *Ans.* $\frac{a^6-x^6}{a^2-x^2}$.

6. Reduce $a+b+c$ to a fraction, having $a+b-c$ for its denominator. *Ans.* $\frac{(a+b)^2-c^2}{a+b-c}$.

PROBLEM.

(154.) To convert a fraction into an equivalent one having a given denominator.

RULE.

Divide the given denominator by the denominator of the fraction, and multiply both terms of the fraction by the quotient.

PROBLEM.

(155.) Convert $\frac{a}{b}$ into a fraction, having ab for its denominator.

SOLUTION.

Dividing ab by b, and multiplying both terms of $\frac{a}{b}$ by the quotient a, gives $\frac{a^2}{ab}$ for the required form.

EXAMPLES.

1. Convert $\frac{2}{3}$ into a fraction, having 6 for its denominator.

Ans. $\frac{4}{6}$.

2. Convert $\frac{7}{8}$ into a fraction, having 32 for its denominator.

Ans. $\frac{28}{32}$.

3. Convert $\frac{x+y}{x-y}$ into a fraction, having x^2-y^2 for its denominator.

Ans. $\frac{(x+y)^2}{x^2-y^2}$.

4. Convert $\frac{a+b}{a^2-ab+b^2}$ into a fraction having a^3+b^3 for its denominator.

Ans. $\frac{(a+b)^2}{a^3+b^3}$.

5. Convert $\frac{a^2+x^2}{a^4+a^2x^2+x^4}$ into a fraction having a^6-x^6 for its denominator.

Ans. $\frac{a^4-x^4}{a^6-x^6}$.

ADDITION OF FRACTIONS.

PROBLEM.

(156.) To find the sum of two or more algebraic fractions.

RULE.

Reduce the fractions to a common denominator, and add together the numerators, and write the sum over the common denominator.

PROBLEM.

(157.) Add $\frac{3a^2}{2b}$, $\frac{2a}{5}$, and $\frac{b}{7}$ together.

SOLUTION.

Performing the operations indicated by the rule, we have

$$\frac{3a^2}{2b}+\frac{2a}{5}+\frac{b}{7}=\frac{105a^2}{70b}+\frac{28ab}{70b}+\frac{10b^2}{70b}=\frac{105a^2+28ab+10b^2}{70b}.$$

EXAMPLES.

1. Add $\frac{a}{b}$, $\frac{2a}{3b}$, and $\frac{5b}{4a}$ together. *Ans.* $\frac{20a^2+15b^2}{12ab}$.

2. Add $\frac{a+b}{a-b}$ and $\frac{a-b}{a+b}$ together. *Ans.* $\frac{2(a^2+b^2)}{a^2-b^2}$.

3. Add $\frac{a}{b}$ and $\frac{c}{d}$ together. *Ans.* $\frac{ad+bc}{bd}$.

4. Add $\frac{x}{x+3}$ and $\frac{x}{x-3}$ together. *Ans.* $\frac{2x^2}{x^2-9}$.

5. Add $\frac{x}{2}$, $\frac{x}{3}$, and $\frac{x}{4}$ together. *Ans.* $x+\frac{x}{12}$.

6. Add $4a$, $5a+\frac{2x}{5}$, and $a-\frac{8x}{9}$ together. *Ans.* $10a-\frac{22x}{45}$.

7. Add $3c+\frac{3x}{5}$, $\frac{a}{a-x}$, and $\frac{a-x}{a}$ together.

Ans. $3c+2+\frac{3a^2x-3ax^2+5x^2}{5a(a-x)}$.

8. Add $7x+\frac{x-2}{3}$ and $9x-\frac{2x-3}{5x}$ together.

Ans. $16x+\frac{5x^2-16x+9}{15x}$.

9. Add $\frac{a+x}{a-x}$ and $-\frac{a-x}{a+x}$ together. *Ans.* $\frac{4ax}{a^2-x^2}$.

10. Add $5x-\frac{2x}{7}$ and $\frac{5x}{9}-4x$ together. *Ans.* $x+\frac{17x}{63}$.

11. Add $\frac{a}{a-x}$ and $\frac{a-x}{a}$ together. *Ans.* $2+\frac{x^2}{a^2-ax}$.

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12. Add $\frac{1}{1+x}$ and $\frac{1}{1-x}$ together. *Ans.* $\frac{2}{1-x^2}$.

13. Add $\frac{1+x^2}{1-x^2}$ and $\frac{1-x^2}{1+x^2}$ together. *Ans.* $\frac{2(1+x^4)}{1-x^4}$.

14. Add together $\frac{2a}{b}$, $\frac{3a^2}{6}$, $\frac{2b}{a}$, and $\frac{1}{2}$. *Ans.* $\frac{4a^2+a^3b+4b^2+ab}{2ab}$.

15. Add $\frac{x}{x^2-y^2}$, $\frac{y}{x+y}$, and $\frac{1}{x-y}$ together. *Ans.* $\frac{2x+xy-y^2+y}{x^2-y^2}$.

16. Add together $\frac{a}{3my^2-x}$ and $\frac{y-6amy^2}{(3my^2-x)^2}$.

Ans. $\frac{y-3amy^2-ax}{(3my^2-x)^2}$.

17. Add together $\frac{2x}{3}$, $\frac{7x}{4}$, and $\frac{2x+1}{5}$. *Ans.* $2x+\frac{1}{5}+\frac{49x}{60}$.

18. Add together $\frac{a^3}{(a+b)^3}$, $\frac{b}{a+b}$, and $-\frac{ab}{(a+b)^2}$.

Ans. $\frac{a^3+ab^2+b^3}{(a+b)^3}$.

19. Add together $\frac{3a}{(a-2x)^2}$, $\frac{2a+x}{(a+x)(a-2x)}$, and $-\frac{5}{a+x}$.

Ans. $\frac{2(10a-11x)x}{(a+x)(a-2x)^2}$.

20. Add together $\frac{3}{4(1-x)^2}$, $\frac{3}{8(1-x)}$, $\frac{1}{8(1+x)}$, and $-\frac{1-x}{4(1+x^2)}$.

Ans. $\frac{1+x+x^2}{1-x-x^4+x^5}$.

SUBTRACTION OF FRACTIONS.

PROBLEM.

(158.) To find the difference of two algebraic fractions.

RULE.

Reduce the fractions to a common denominator, and subtract the numerator of the fraction to be subtracted from the numerator of the other fraction, and place the result over the common denominator.

PROBLEM.

(159.) Subtract $\frac{1+x}{1-x}$ from $\frac{1-x}{1+x}$.

SOLUTION.

Performing the operation indicated in the rule, we have

$$\frac{1-2x+x^2}{1-x^2}-\frac{1+2x+x^2}{1-x^2}=\frac{-4x}{1-x^2}=-\frac{4x}{1-x^2}.$$

EXAMPLES.

1. From $\frac{a}{b}$ subtract $\frac{c}{d}$. *Ans.* $\frac{ad-bc}{bd}$.

2. From $\frac{a+b}{a-b}$ subtract $\frac{a-b}{a+b}$. *Ans.* $\frac{4ab}{a^2-b^2}$.

3. From $\frac{1+x^2}{1-x^2}$ subtract $\frac{1-x^2}{1+x^2}$. *Ans.* $\frac{4x^2}{1-x^4}$.

4. From $\frac{1}{a-x}$ subtract $\frac{1}{a+x}$. *Ans.* $\frac{2x}{a^2-x^2}$.

5. From $3x+\frac{x}{b}$ subtract $x-\frac{x-a}{c}$. *Ans.* $2x+\frac{cx+bx-ab}{bc}$.

6. From $5x+\frac{x-2}{3}$ subtract $4x-\frac{2x-3}{5}$. *Ans.* $x+\frac{11x-19}{15}$.

7. From $\frac{a-x}{a(a+x)}+a$ subtract $\frac{a+x}{a(a-x)}$. *Ans.* $a-\frac{4x}{a^2-x^2}$.

8. From $3x$ subtract $\frac{3a+12x}{5}$. *Ans.* $\frac{3x-3a}{5}$.

9. From $2x+\frac{5x-2}{7}$ subtract $3x-\frac{4x+5}{6}$. *Ans.* $\frac{16x+23}{42}$.

10. From $ax+\frac{2x+7}{8}$ subtract $x-\frac{5x-6}{21}$. *Ans.* $ax-\frac{86x-99}{168}$.

11. From $\frac{12x}{7}$ subtract $\frac{3x}{5}$. *Ans.* $x+\frac{4x}{35}$.

12. From $9y$ subtract $\frac{1+2y}{8}$. *Ans.* $8y+\frac{6y-1}{8}$.

13. From $\frac{x}{x-3}$ subtract $\frac{x+3}{x}$. *Ans.* $\frac{9}{x^2-3x}$.

14. From $\frac{1}{a-b}$ subtract $\frac{1}{a+b}$. *Ans.* $\frac{2b}{b^2-a^2}$. ? $\frac{2b}{a^2-}$

15. From $\frac{a}{b}+\frac{m}{n}$ subtract $\frac{p}{q}+\frac{x}{y}$.

Ans. $\frac{anqy+mbqy-pbny-xbnq}{bnqy}$.

16. From $\frac{a-b}{ab}+\frac{b-c}{bc}$ subtract $\frac{a-c}{ac}$. *Ans* 0.

17. From $\frac{x+y}{y}$ subtract $\frac{x}{x+y}+\frac{x^3-x^2y}{x^2y-y^3}$. *Ans.* 1.

18. From $\frac{4x-3y}{3(1-y)}+\frac{2y}{1-y}$ subtract $\frac{x+3y}{3(1-y)}$. *Ans.* $\frac{x}{1-y}$.

19. From $\frac{1}{x-1}$ subtract $\frac{1}{2(x+1)}+\frac{x+3}{2(x^2+1)}$. *Ans.* $\frac{x+3}{x^4-1}$.

20. From $\frac{1}{x^3}+\frac{1}{x^2}+\frac{x-1}{x^2+1}$ subtract $\frac{1}{x}+\frac{1}{(x^2+1)^2}$.

Ans. $\frac{x^2+x+1}{x^3(x^2+1)^2}$.

MULTIPLICATION OF FRACTIONS.

PROBLEM.

(160.) To find the product of two or more algebraic fractions.

RULE.

Multiply the numerators together for a new numerator, and the denominators together for a new denominator.

DEMONSTRATION.

Let $\frac{A}{B}$ and $\frac{C}{D}$ represent any two fractions.

We are here to prove that $\frac{A}{B}\times\frac{C}{D}=\frac{AC}{BD}$.

Since $\frac{A}{B}=AB^{-1}$ and $\frac{C}{D}=CD^{-1}$, $\frac{A}{B}\times\frac{C}{D}$ must equal $AB^{-1}\times CD^{-4}$. But by Prop. 1 in Multiplication, we have $AB^{-1}\times CD^{-1}=AB^{-1}CD^{-1}=\frac{AB^{-1}CD^{-1}}{1}$ which, freed from negative exponents by Prop. **(101)**, is $\frac{AC}{BB}$. Q. E. D.

PROBLEM.

(161.) Find the product of $\frac{x^2+3x+2}{x^2+2x+1}$ and $\frac{x^2+5x+4}{x^2+7x+12}$.

SOLUTION.

To save labor, let us resolve the terms in both of the fractions into their simplest factors, and proceed as directed by the rule, merely indicating the multiplication, and then canceling the factors $x+1$, $x+1$, and $x+4$, common to both terms, we have

$$\frac{(x+1)(x+2)(x+1)(x+4)}{(x+1)(x+1)(x+3)(x+4)}=\frac{x+2}{x+3}.$$

EXAMPLES.

1. Find the product of $\frac{2x}{5}$ and $\frac{3x^2}{2a}$ *Ans.* $\frac{3x^3}{5a}$.

2. Find the product of $\frac{2x}{a}$, $\frac{3ab}{c}$, and $\frac{5ac}{2b}$. *Ans.* $15ax$.

3. Find the product of $\frac{2x}{3}$, $\frac{4x^2}{7}$, and $\frac{a}{a+x}$. *Ans.* $\frac{8ax^3}{21(a+x)}$.

4. Find the product of $3x$, $\frac{x+1}{2a}$, and $\frac{x-1}{a+b}$. *Ans.* $\frac{3x(x^2-1)}{2a(a+b)}$.

5. Find the product of $2a+\frac{bx}{a}$ and $3a-\frac{b}{ax}$.

Ans. $6a^2+3bx-\frac{2b}{x}-\frac{b^2}{a^2}$.

6. Find the product of $\frac{a^2-x^2}{a+b}$, $\frac{a^2-b^2}{ax+x^2}$, and $a+\frac{ax}{a-x}$.

Ans. $\frac{a^2(a-b)}{x}$.

7. Find the product of $\frac{a^2+b^2}{a^2-b^2}$ and $\frac{a-b}{a+b}$. Ans. $\frac{a^2+b^2}{(a+b)^2}$.

8. Find the product of $\frac{x^2-9x+20}{x^2-6x}$ and $\frac{x^2-13x+42}{x^2-5x}$.

Ans. $\frac{x^2-11x+28}{x^2}$:

9. Find the product of $\frac{a^2-x^2}{3a}$ and $\frac{7x^2}{a-x}$. Ans. $\frac{7(a+x)x^2}{3a}$.

10. Find the product of $a+\frac{x}{5}$ and $a-\frac{x}{3}$ Ans. $a^2-\frac{(2a+x)x}{15}$.

11. Find the product of $\frac{3x^2}{5x-10}$ and $\frac{15x-30}{2x}$ Ans. $4x+\frac{x}{2}$.

12. Find the product of $\frac{2a-2x}{3ab}$ and $\frac{3ax}{5a-5x}$. Ans. $\frac{2x}{5b}$.

13. Find the product of $\frac{a^2-x^2}{a+b}$, $\frac{a^2-b^2}{a+x}$, and $\frac{a}{ax-x^2}$.

Ans. $\frac{a(a-b)}{x}$.

14. Find the product of $\frac{a^4-x^4}{a^2-y^2}$, $\frac{a+y}{a^2+x^2}$, and $\frac{a-y}{a-x}$. Ans. $a+x$.

15. Find the product of $a+\frac{ax}{a-x}$ and $x-\frac{ax}{a+x}$. Ans. $\frac{a^2x^2}{a^2-x^2}$.

16. Find the product of $\frac{x^2-b^2}{bc}$ and $\frac{x^2+b^2}{b+c}$. Ans. $\frac{x^4-b^4}{bc(b+c)}$.

17. Find the product of $\frac{5a}{b}-\frac{13c}{2d}-\frac{6h}{5bg}+7d$ and $\frac{3a}{5d}$.

Ans. $\frac{3a^2}{bd}-\frac{39ac}{10d^2}-\frac{18ah}{25bdg}+\frac{21a}{5}$.

18. Find the product of $\frac{a^2}{x^2}-\frac{ab}{2xy}+\frac{b^2}{y^2}$ and $\frac{3a^2}{x^2}-\frac{2ab}{5xy}+\frac{b^2}{y^2}$.

Ans. $\frac{3a^4}{x^4}-\frac{19a^3b}{10x^3y}+\frac{21a^2b^2}{5x^2y^2}-\frac{9ab^3}{10xy^3}+\frac{b^4}{y^4}$.

19. Find the product of $\frac{3(a^2-x^2)+a-x}{2}$ and $\frac{4}{3(a-x)}$.

Ans. $2(a+x)+\frac{2}{3}$.

20. Find the product of $\frac{4a^2-16b^2}{a-2b}$ and $\frac{5a}{20a^2+80ab+80b^2}$.

Ans. $\frac{a}{a+2b}$.

DIVISION OF FRACTIONS.

PROBLEM.

(162.) To divide one fraction by another.

RULE.

Invert the terms of the divisor, and proceed as in Multiplication.

DEMONSTRATION.

Let $\frac{A}{B}$ and $\frac{C}{D}$ represent any two fractions.

We are to prove that $\frac{A}{B}\div\frac{C}{D}=\frac{A}{B}\times\frac{D}{C}=\frac{AD}{BC}$.

Since $\frac{A}{B}=AB^{-1}$ and $\frac{C}{D}=CD^{-1}$, $\frac{A}{B}\div\frac{C}{D}$ must equal $AB^{-1}\div CD^{-1}$, or $\frac{AB^{-1}}{CD^{-1}}$, which, freed from negative exponents, is $\frac{AD}{BC}$. *Q. E. D.*

PROBLEM.

(163.) Divide $\frac{2x^2}{a^3+x^3}$ by $\frac{x}{x+a}$.

SOLUTION.

By the rule, we have

$$\frac{2x^2}{a^3+x^3}\times\frac{x+a}{x}=\frac{2x^2(x+a)}{x(a^3+x^3)}=\frac{2x^2(x+a)}{x(a+x)(a^2-ax+x^2)}=\frac{2x}{a^2-ax+x^2}.$$

EXAMPLES.

1. Divide $\frac{x+a}{2x-2b}$ by $\frac{x+b}{5x+a}$. *Ans.* $\frac{5x^2+6ax+a^2}{2(x^2-b^2)}$.

2. Divide $\frac{x}{3}$ by $\frac{2x}{9}$. *Ans.* $1\frac{1}{2}$.

3. Divide $\frac{3x^2-3x}{5}$ by $\frac{x^2}{5}$. *Ans.* $\frac{3(x-1)}{x}$.

4. Divide $\frac{4x+2}{5}$ by $\frac{2x+1}{5x}$. *Ans.* $2x$.

5. Divide $\frac{a^3-x^3}{a+x}$ by $\frac{a-x}{a^2+2ax+x^2}$. *Ans.* $a^3+2a^2x+2ax^2+x^3$.

6. Divide $\frac{a}{b}+\frac{c}{d}$ by $\frac{e}{f}+\frac{g}{h}$. *Ans.* $\frac{(ad+bc)fh}{(eh+fg)bd}$.

7. Divide $\frac{x+a}{x-b}$ by $\frac{x+b}{5x+a}$ *Ans.* $\frac{5x^2+6ax+a^2}{x^2-b^2}$.

8. Divide $\frac{2ax+x^2}{c^3-x^3}$ by $\frac{x}{c-x}$ *Ans.* $\frac{2a+x}{c^2+cx+x^2}$.

9. Divide $\frac{a}{a+b}+\frac{b}{a-b}$ by $\frac{a}{a-b}-\frac{b}{a+b}$. *Ans.* 1.

10. Divide $\frac{x^4-b^4}{x^2-2bx+b^2}$ by $\frac{x^2+bx}{x-b}$. *Ans.* $x+\frac{b^2}{x}$.

11. Divide $\frac{x^2-a^2}{a+c}$ by $a+\frac{x^2-a^2}{a}$. *Ans.* $\frac{a(x^2-a^2)}{x^2(a+c)}$.

12. Divide $\frac{3a-3x}{a+b}$ by $\frac{5a-5x}{a+b}$. *Ans.* $\frac{3}{5}$.

13. Divide 12 by $\frac{(a+x)^2}{x}-a$. *Ans.* $\frac{12x}{a^2+ax+x^2}$.

14. Divide $\frac{a^4-2a^2x^2+x^4}{a^3x+ax^3}$ by $\frac{a^2-x^2}{a^2+x^2}$. *Ans.* $\frac{a^2-x^2}{ax}$.

15. Divide $1+\frac{n-1}{n+1}$ by $1-\frac{n-1}{n+1}$. *Ans.* n.

16. Divide $a+\frac{2ax-1}{b}$ by $\frac{x-a}{ax+1}$.

Ans. $\frac{a^2(bx+2x^2)+a(x+b)-1}{b(x-a)}$.

17. Divide $\frac{a+x}{a-x}+\frac{a-x}{a+x}$ by $\frac{a+x}{a-x}-\frac{a-x}{a+x}$. *Ans.* $\frac{a^2+x^2}{2ax}$.

18. Divide $\frac{a^6-b^6}{a^2-2ab+b^2}$ by $\frac{a^2+ab+b^2}{a-b}$. *Ans.* a^3+b^3.

19. Divide $\frac{a^3c}{b^5}+\frac{a^4c}{b^4}-\frac{7a^5c}{b^3}-\frac{3a^6c}{b^2}+\frac{a^2c^3}{b^2}-\frac{2a^3c^3}{b}-a^4c^3$ by $\frac{a}{b^3}+\frac{3a^2}{b^2}+c^2$.

Ans. $\frac{a^2c}{b^2}-\frac{2a^3c}{b}-a^4c$.

MISCELLANEOUS PROPOSITIONS.

PROPOSITION

(164.) 1. If the same quantity be added to both terms of a fraction the resulting fraction will be greater or less than the given fraction, according as the numerator of the given fraction is less or greater than the denominator.

DEMONSTRATION

To be supplied by the student.

PROPOSITION

(165.) 2. If the same quantity be subtracted from both terms of a fraction, the resulting fraction will be less or greater than the given fraction, according as the numerator of the given fraction is less or greater than the denominator.

DEMONSTRATION

To be supplied by the student.

PROPOSITION

(166.) 3. If two fractions added together equal unity, their difference is equal to the difference of their squares.

DEMONSTRATION

To be supplied by the student.

PROPOSITION

(167.) 4. The sum or difference of two quantities divided by their product, is equal to the sum or difference of their reciprocals.

DEMONSTRATION

To be supplied by the student.

PROPOSITION

(168.) 5. If the difference of two quantities is equal to $\frac{x}{y}$, their sum multiplied by x equals the difference of their squares multiplied by y.

DEMONSTRATION

To be supplied by the student.

PROPOSITION

(169.) 6. Zero divided by a finite quantity equals zero.

DEMONSTRATION.

Let b represent any finite quantity.

We are to prove that $\frac{0}{b}=0$.

It is evident that the value of the expression $\frac{a}{b}$ will become less by diminishing a when b remains constant. Therefore, when a is assumed to be less than any assignable quantity, the value of the expression $\frac{a}{b}$ must also be less than any assignable quantity, or in other words, when $a=0$, we have $\frac{0}{b}=0$. *Q. E. D.*

PROPOSITION

(170.) 7. A finite quantity divided by zero equals infinity.

DEMONSTRATION.

Let a represent any finite quantity.

We are to prove that $\frac{a}{0}=\infty$.

It is evident that the value of the expression $\frac{a}{b}$ will become greater by diminishing b, when a remains constant. Therefore, when b is assumed to be less than any assignable quantity, the value of the expression $\frac{a}{b}$ must be greater than any assignable quantity, or in other words, when $b=0$, we have $\frac{a}{0}=\infty$. *Q. E. D.*

PROPOSITION

(171.) 8. Zero divided by zero, considered without reference to the expression from which it is derived, may represent any value whatever.

DEMONSTRATION.

Since, $a \times 0 = 0$, by making the last zero the dividend, and the other the divisor, we have

$$\frac{0}{0} = a$$

Since a may be of any value whatever, the truth of the proposition is established.

VANISHING FRACTIONS.

(172.) A vanishing fraction is one which becomes equal to $\frac{0}{0}$ when certain suppositions are made.

PROBLEM.

(173.) To find the value of a vanishing fraction.

SOLUTION.

Let $\frac{x^m - y^m}{x - y}$ be a fraction whose value is sought when it becomes equal to $\frac{0}{0}$. By **(112)**, we have

$$\frac{x^m - y^m}{x - y} = x^{m-1} + x^{m-2}y + x^{m-3}y^2 \ldots\ldots + x^2y^{m-3} + xy^{m-2} + y^{m-1}.$$

If now we make $y = x$, we have

$$\frac{x^m - x^m}{x - x} = x^{m-1} + x^{m-1} + x^{m-1} \ldots\ldots + x^{m-1} + x^{m-1} + x^{m-1}.$$

Since there must be m terms in the quotient, each of which equals x^{m-1}, the whole quotient must be mx^{m-1}. Hence, we have

$$\frac{x^m - y^m}{x - y} = \frac{0}{0} = mx^{m-1}, \text{ when } y = x.$$

In this example it may be seen that we first obtained an expression for the value of the given fraction, before making the supposition

which reduced it to a vanishing fraction, and it was this process that enabled us to find the value of the given fraction when it became $\frac{0}{0}$.

Hence, to find the value of a vanishing fraction the following

RULE.

Find an expression for the value of the given fraction, and then make the supposition necessary to reduce the given fraction to $\frac{0}{0}$.

EXAMPLES.

1. Find the value of $\frac{a^2-b^2}{a-b}$ when $b=a$. *Ans.* $2a$.

2. Find the value of $\frac{x^3-a^3}{x-a}$ when $x=a$. *Ans.* $3a^2$.

3. Find the value of $\frac{x^4-a^4}{x-a}$ when $x=a$. *Ans.* $4a^3$.

4. Find the value of $\frac{x^4-a^4}{x^2-a^2}$ when $x=a$. *Ans.* $2a^2$

5. Find the value of $\frac{x^3-a^3}{x^2-a^2}$ when $x=a$. *Ans.* $\frac{3a}{2}$.

6. Find the value of $\frac{x-a^{\frac{1}{2}}x^{\frac{1}{2}}}{x-a}$ when $x=a$. *Ans.* $\frac{1}{2}$.

7. Find the value of $\frac{x^2-a^2}{(x-a)^2}$ when $x=a$. *Ans.* ∞.

8. Find the value of $\frac{(x-a)^2}{x^3-a^3}$ when $x=a$. *Ans.* 0.

9. Find the value of $\frac{(x^2-a^2)^{\frac{3}{2}}}{(x-a)^{\frac{3}{2}}}$ when $x=a$. *Ans.* $(2a)^{\frac{3}{2}}$.

10. Find the value of $\frac{x-x^5}{1-x}$ when $x=1$. *Ans.* 4.

11. Find the value of $\frac{x^m-a^m}{x-a}$ when $x=a$. *Ans.* ma^{m-1}.

12. Find the value of $\frac{1-x^n}{1-x}$ when $x=1$. *Ans.* n.

13. Find the value of $\dfrac{x^4+ax^3-9a^2x^2+11a^3x-4a^4}{x^4-ax^3-3a^2x^2+5a^3x-2a^4}$ when $x=a$.

Ans. $\dfrac{5}{3}$.

14. Find the value of $\dfrac{3a^3-10a^2x+4ax^2+8x^3}{a^2-4x^2}$ when $x=\frac{1}{2}a$.

Ans. 0.

15. Find the value of $\dfrac{3a^2-4ax-4x^2}{a^3-2a^2x-4ax^2+8x^3}$ when $x=\frac{1}{2}a$.

Ans. ∞.

16. Find the value of $\dfrac{a^1a^{\frac{1}{2}}x^{\frac{1}{2}}-x^2}{a-a^{\frac{1}{2}}x^{\frac{1}{2}}}$ when $x=a$. *Ans.* $3a$.

17. Find the value of $\dfrac{nx^{n+1}-(n+1)x^n+1}{1-x^2}$ when $x=1$. *Ans.* 0

18. Find the value of $\dfrac{x^{\frac{1}{2}}-a^{\frac{1}{2}}+(x-a)^{\frac{1}{2}}}{(x^2-a^2)^{\frac{1}{2}}}$ when $x=a$.

Ans. $2^{-\frac{1}{2}}a^{-\frac{1}{2}}$.

19. Find the value of $\dfrac{(x-1)^{\frac{1}{2}}+2(x^2-1)^{\frac{1}{2}}}{(x^3-1)^{\frac{1}{2}}}$ when $x=1$.

Ans. $\dfrac{1+2(2)^{\frac{1}{2}}}{3^{\frac{1}{2}}}$.

20. Find the value of $\dfrac{(x^2+x-2)^{\frac{1}{2}}+(2x^3-x-1)^{\frac{2}{3}}}{(x^5-2x^4+x^3+x^2-2x+1)^{\frac{1}{4}}}$ when $x=1$.

Ans. $\dfrac{3^{\frac{1}{2}}}{2^{\frac{1}{4}}}$.

21. Find the value of $\dfrac{ax^2+ac^2-2acx}{bx^2-2bcx+bc^2}$ when $x=c$. *Ans.* $\dfrac{a}{b}$.

22. Find the value of $\dfrac{x^3-ax^2-a^2x+a^3}{x^2-a^2}$ when $x=a$. *Ans.* 0.

23. Find the value of $\dfrac{ax-x^2}{a^4-2a^3x+2ax^3-x^4}$ when $x=a$. *Ans.* ∞.

24. Find the value of $\dfrac{a-(a^2-x^2)^{\frac{1}{2}}}{x^2}$ when $x=0$. *Ans.* $\dfrac{1}{2a}$.

CHAPTER VII.

INVOLUTION.

(174.) Involution is raising a given quantity to a given power.

PROBLEM.

(175.) To raise a monomial to the nth power.

RULE.

Multiply all the exponents contained in the monomial by the index of the power, and prefix $+$ *to the result, except when the monomial is negative and the index of the power is odd, in which case prefix* $-$.

DEMONSTRATION.

Let $3ab^2c^3$ be a given monomial whose nth power is sought. We know by the definition of a power that the nth power of $3ab^2c^3$ equals $3ab^2c^3$ taken n times as a factor, or, in other words, equals the product of 3^1, a^1, b^2, and c^3, each taken n times as a factor. Therefore, $(3^1a^1b^2c^3)^n=3^na^n(b^2)^n(c^3)^n$.

Since b^2 taken n times as a factor is the same as b taken $2n$ times as a factor, we have $(b^2)^n=b^{2n}$. In the same way we get $(c^3)^n=c^{3n}$, whence $(3ab^2c^3)^n=3^na^nb^{2n}c^{3n}$, which result agrees with the rule. When a positive monomial is raised to any power, it is evident that the result must be positive, since, any number of positive factors multiplied together will produce a positive product; also, when a negative monomial is raised to an even power the result must be positive, since, an even number of negative factors multiplied together will produce a positive product. But, when a negative monomial is raised to an odd power, the result must be negative, since, an odd number of negative factors multiplied together will produce a negative product.

PROBLEM

(176.) 1. Involve $2a^{\frac{1}{2}}b^{\frac{1}{4}}c^n$ to the 4th power.

SOLUTION.

By the rule, we have $2^4a^{\frac{4}{2}}b^{\frac{4}{4}}c^{4m}=16a^2bc^{4m}$. Whenever one of the factors of the monomial is numerical, its power may be found arithmetically. Thus, $2^4=16$.

PROBLEM

2. Raise $\sqrt[n]{a^m}$ to the pth power.

SOLUTION.

Put $\sqrt[n]{a^m}=R$ and $a^m=A$. Since, by **(22)** and **(24)**, we have $\sqrt[n]{a^m}=(a^m)^{\frac{1}{n}}$, we may write $R=(A)^{\frac{1}{n}}$ or $R=A^{\frac{1}{n}}$. By applying the rule, we obtain R^p for the pth power of R^1, and $A^{\frac{p}{n}}$ for the pth power of $A^{\frac{1}{n}}$; therefore, $R^p=A^{\frac{p}{n}}$, or $R^p=(A^p)^{\frac{1}{n}}=\sqrt[n]{A^p}$. Since $R=\sqrt[n]{a^m}=\sqrt[n]{A}$, we have $R^p=(\sqrt[n]{A})^p$; but $R^p=\sqrt[n]{A^p}$, therefore, $(\sqrt[n]{A})^p=\sqrt[n]{A^p}$, which expression, because $A=a^m$ and $A^p=a^{mp}$, becomes $(\sqrt[n]{a^m})^p=\sqrt[n]{a^{mp}}$.

Hence, to raise a radical to any power, we have only to raise the quantity under the radical to that power.

This principle gives $(\sqrt{a})^2=\sqrt{a^2}$, $(\sqrt{a^2})^3=\sqrt{a^6}$, $(\sqrt[3]{a})^2=\sqrt[3]{a^2}$, $(\sqrt{a^{\frac{1}{2}}})^2=\sqrt{a}$, $(\sqrt{a^{\frac{1}{3}}})^3=\sqrt{a}$, $(\sqrt{a^{\frac{2}{3}}})^3=\sqrt{a^2}$, &c.

Since $\sqrt[n]{A}=A^{\frac{1}{n}}$, we have $(\sqrt[n]{A})^p=A^{\frac{p}{n}}$. If, now, we suppose $p=n$, we obtain $(\sqrt[n]{A})^n=A^{\frac{n}{n}}=A^1=A$, which expression, because $A=a^m$, becomes $(\sqrt[n]{a^m})^n=a^m$.

Hence, to raise a radical to a power equal to the index of the root, we have only to remove the radical.

This principle gives $(\sqrt{a})^2=a$, $(\sqrt{a^3})^2=a^3$, $(\sqrt{-a})^2=-a$, $(\sqrt[3]{-1})^3=-1$, $(\sqrt[3]{a^2})^3=a^2$, $\left(\sqrt[n]{-2^{\frac{1}{n}}}\right)^n=-2^{\frac{1}{n}}$, &c.

PROBLEM

3. Raise $\sqrt{-a}$ to the third power.

SOLUTION.

According to the first principle given in the last solution, we have $(\sqrt{-a})^3=\sqrt{-a^3}$; but the cube of any quantity is the product ob-

tained by taking the quantity three times as a factor; therefore, $(\sqrt{-a})^3=\sqrt{-a}\,.\sqrt{-a}\,.\sqrt{-a}$. Since, according to the second principle, $\sqrt{-a}\,.\sqrt{-a}$, or $(\sqrt{-a})^2=-a$, we have $\sqrt{-a}\,.\sqrt{-a}\,.\sqrt{-a}=-a\sqrt{-a}$; whence, we get $(\sqrt{-a})^3=-a\sqrt{-a}$. We have now obtained two expressions for the cube of $\sqrt{-a}$; one being $\sqrt{-a^3}$, and the other $-a\sqrt{-a}$; hence, in this particular case, $\sqrt{-a^3}=-a\sqrt{-a}$. We say in this particular case, because $\sqrt{-a^3}=+a\sqrt{-a}$, when $-a^3$ has been obtained by multiplying $-a$ by $a^2=+a\,.+a$; for, when $\sqrt{-a^3}=\sqrt{a\,.\,a\,.-a}=\sqrt{a}\,.\sqrt{a}\,.\sqrt{-a}$, we have $\sqrt{-a^3}=a\sqrt{-a}$, since, according to the second principle, $\sqrt{a}\,.\sqrt{a}=a$.

Let us suppose that $a=1$ in this problem, or that we seek the cube of $\sqrt{-1}$. According to the first principle we have $(\sqrt{-1})^3=\sqrt{-1}$, but according to the second, we get $(\sqrt{-1})^3=-1\sqrt{-1}=-\sqrt{-1}$. But how can $\sqrt{-1}=-\sqrt{-1}$? From the nature of the problem, we know the $\sqrt{-1}$ which we have put equal to $(\sqrt{-1})^3$ is equal to $\sqrt{-1^3}=\sqrt{-1.-1.-1}=\sqrt{-1}.\sqrt{-1}.\sqrt{-1}$ which equals $-1\sqrt{-1}$, or $-\sqrt{-1}$, because, by the second principle $\sqrt{-1}.\sqrt{-1}$, or $(\sqrt{-1})^2$, is equal to -1.

The result $\sqrt{-1}$ is not a convenient form for the cube of $\sqrt{-1}$, since, the composition of the -1 is not apparent, because it may be the product of -1, $+1$, and $+1$, as well as -1, -1, and -1.

Hence, we have $\sqrt{-1}=+\sqrt{-1}$ or $-\sqrt{-1}$, according as the -1 is considered as being composed of the factors -1, $+1$, and $+1$, or of the factors -1, -1, and -1.

By using the second principle, we shall always arrive at results which are not ambiguous. We may, however, arrive at equally definite results by the first principle whenever the quantity under the radical is positive, but the results will not always be of the simplest form.

EXAMPLES.

1. Square $3a^3b$. *Ans.* $9a^6b^2$.

2. Square $7ax$. *Ans.* $49a^2x^2$.

3. Cube $2a^2bc^3$. *Ans.* $8a^6b^3c^9$.

4. Cube $-a^{\frac{1}{2}}b^{\frac{1}{3}}c$. *Ans.* $-a^{\frac{3}{2}}bc^3$.

5. Raise $x^{\frac{1}{2}}y^{-\frac{1}{2}}$ to the 4th power. *Ans.* x^2y^{-2}.

6. Raise $-a^{-2}x^{-\frac{1}{2}}$ to the 7th power. *Ans.* $-\dfrac{1}{a^{14}x^{\frac{7}{2}}}$.

7. Raise $3a^2x^3$ to the nth power. *Ans.* $3^na^{2n}x^{3n}$.

8. Raise $-a^m$ to the nth power, n being even. *Ans.* a^{mn}.

9. Raise $-a^m$ to the nth power, n being odd. *Ans.* $-a^{mn}$.

10. Raise $7a^mb_1^nc^3$ to the 0th power. *Ans.* 1.

11. Raise $\sqrt{a+b}$ to the 2nd power. *Ans.* $a+b$.

12. Raise $\sqrt{a+b}$ to the 3rd power. *Ans.* $(a+b)\sqrt{a+b}$.

13. Raise $\sqrt{a-x}$ to the 4th power. *Ans.* $(a-x)^2$.

14. Raise $a\sqrt[3]{a+x}$ to the 3rd power. *Ans.* a^4+a^3x.

15. Raise $a\sqrt{a}$ to the 5th power. *Ans.* $a^7\sqrt{a}$.

16. Raise $-a\sqrt{-a}$ to the 5th power. *Ans.* $-a^7\sqrt{-a}$.

17. Raise $\sqrt{-1}$ to the 6th power. *Ans.* -1.

18. Raise $-\sqrt{-1}$ to the 3rd power. *Ans.* $\sqrt{-1}$.

19. Raise $-\sqrt{a}$ to the 3rd power. *Ans.* $-a\sqrt{a}$.

20. Raise $\sqrt{-a}\,.\sqrt{-1}$ to the 3rd power. $a\sqrt{a}$ *Ans.* $-a\sqrt{a}$.

BINOMIAL THEOREM.

(177.) The nth power of a binomial is represented by the following formula: $(x+y)^n = x^n + nx^{n-1}y + n\,.\,\frac{n-1}{2}x^{n-2}y^2 + n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}$ $x^{n-3}y^3 + \ldots\ldots\ldots\ldots + n\,.\,\frac{n-1}{2}x^2y^{n-2} + xy^{n-1} + y^n$.

DEMONSTRATION.

A full demonstration of this celebrated Theorem will be given in the chapter on Series.

An inspection of the above formula shows that the exponents of x commence with the index of the power, and decrease by unity, and

that the exponents of y commencing in the second term increase by unity. Thus, for the literal part of the 5th power of $x+y$, we have $x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5$.

Let us now observe the law of the coefficients in the above formula. We see that the coefficient of the first term is unity, and the coefficient of the second term is equal to the index of the power, and the coefficient of the third term is equal to the product of the coefficient of the second term by the quotient obtained by dividing the exponent of x in the second term by one more than the exponent of y in the second term, and in general, any coefficient is equal to the product of the coefficient of the preceding term by the quotient obtained by dividing the exponent of x in this preceding term by one more than the exponent of y in the same term.

Thus, we have $(x+y)^5=x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$.

Here the coefficient of the third term is obtained from $5x^4y$ by dividing the exponent of x, which is 4, by one more than the exponent of y, and multiplying the quotient by 5, thus $5\,.\,\frac{4}{2}=10$. For the coefficient of the fourth term, we have $10\,.\,\frac{3}{3}=10$; for that of the fifth, $10\,.\,\frac{2}{4}=5$; and for that of the sixth, $5\,.\,\frac{1}{5}=1$.

It may be observed that after the middle term the coefficients recur in reverse order; and also, that when the second term of the binomial is negative, the terms of any power of it will be alternately positive, and negative, or what is the same, those terms of the power which contain an odd power of y will be negative.

PROBLEM.

(178.) To raise a binomial to a given power.

RULE.

Perform the multiplication indicated, or proceed according to the principles of the binomial theorem.

PROBLEM

(179.) 1. Raise $(2a+3b)$ to the 6^{th} power.

SOLUTION.

By the Binomial Theorem, we have $(2a+3b)^6=(2a)^6+6(2a)^5(3b)+15(2a)^4(3b)^2+20(2a)^3(3b)^3+15(2a)^2(3b)^4+6(2a)(3b)^5+(3b)^6$.

After performing the operations indicated, we obtain $(2a+3b)^6 = 64a^6+576a^5b+2160a^4b^2+4320a^3b^3+4860a^2b^4+2916ab^5+729b^6$.

PROBLEM

2. Cube $\sqrt{2}+\sqrt{3}$.

SOLUTION.

By the Binomial Theorem, we have $(\sqrt{2}+\sqrt{3})^3=(\sqrt{2})^3+3(\sqrt{2})^2(\sqrt{3})+3(\sqrt{2})(\sqrt{3})^2+(\sqrt{3})^3$.

After performing the operations indicated, we obtain $(\sqrt{2}+\sqrt{3})^3= 2\sqrt{2}+6\sqrt{3}+9\sqrt{2}+3\sqrt{3}$.

But $2\sqrt{2}+9\sqrt{2}=11\sqrt{2}$ and $6\sqrt{3}+3\sqrt{3}=9\sqrt{3}$; therefore, $(\sqrt{2}+\sqrt{3})^3 =11\sqrt{2}+9\sqrt{3}$.

EXAMPLES.

1. Raise $a+b$ to the 8th power.

Ans. $a^8+8a^7b+28a^6b^2+56a^5b^3+70a^4b^4+56a^3b^5+28a^2b^6+8ab^7+b^8$.

2. Raise $a-b$ to the 9th power.

Ans. $a^9-9a^8b+36a^7b^2-84a^6b^3+126a^5b^4-126a^4b^5+84a^3b^6-36a^2b^7+9ab^8-b^9$.

3. Raise $5-4x$ to the 4th power.

Ans. $625-2000x+2400x^2-1280x^3+256x^4$.

4. Raise $3-2x^2$ to the 6th power.

Ans. $729-2916x^2+4860x^4-4320x^6+2160x^8-576x^{10}+64x^{12}$.

5. Raise $\frac{1}{2}x+2y$ to the 7th power.

Ans. $\frac{1}{128}x^7+\frac{7}{32}x^6y+\frac{21}{8}x^5y^2+\frac{35}{2}x^4y^3+70x^3y^4+168x^2y^5+224xy^6+128y^7$.

6. Cube $x-\dfrac{1}{x}$. *Ans.* $x^3-\dfrac{1}{x^3}-3\left(x-\dfrac{1}{x}\right)$.

7. Cube $x^{mn}-y^{\frac{1}{2}}$. *Ans.* $x^{3mn}-3x^{2mn}y^{\frac{1}{2}}+3x^{mn}y-y^{\frac{3}{2}}$.

8. Cube $\frac{1}{2}a-\frac{2}{3}b$. *Ans.* $\frac{1}{8}a^3-\frac{8}{27}b^3-\frac{1}{2}a^2b+\frac{2}{3}ab^2$.

9. Cube $\dfrac{a-b}{a-2b}$. *Ans.* $\dfrac{a^3-3a^2b+3ab^2-b^3}{a^3-6a^2b+12ab^2-8b^3}$.

10. Cube x^2+y^2. *Ans.* $x^6+3x^4y^2+3x^2y^4+y^6$.

11. Raise x^2+a^2 to the 4th power.

Ans. $x^8+4x^6a^2+6x^4a^4+4x^2a^6+a^8$.

12. Cube $2a-3b$. *Ans.* $8a^3-36a^2b+54ab^2-27b^3$.

13. Raise $ax-by$ to the 5th power.

Ans. $a^5x^5-5a^4bx^4y+10a^3b^2x^3y^2-10a^2b^3x^2y^3+5ab^4xy^4-b^5y^5$.

14. Cube $2r-6m$. *Ans.* $8r^3-72mr^2+216m^2r-216m^3$.

15. Cube $\sqrt{2}+\sqrt{5}$. *Ans.* $17\sqrt{2}+11\sqrt{5}$.

16. Cube $\sqrt{-1}+\sqrt{-2}$. *Ans.* $-7\sqrt{-1}-5\sqrt{-2}$.

17. Raise $a\sqrt{a}-b\sqrt{b}$ to the 4th power.

Ans. $a^6-4a^4\sqrt{a}b\sqrt{b}+6a^3b^3-4a\sqrt{a}b^4\sqrt{b}+b^6$.

18. Raise $x^{-\frac{3}{4}}-y^{-1}$ to the 4th power.

Ans. $\frac{1}{x^3}-\frac{4}{x^{\frac{9}{4}}y}+\frac{6}{x^{\frac{3}{2}}y^2}-\frac{4}{x^{\frac{3}{4}}y^3}+\frac{1}{y^4}$.

19. Raise $-2\sqrt{-a}-3\sqrt{b}$ to the 4th power.

Ans. $16a^2-96a\sqrt{-a}\sqrt{b}-216ab+216\sqrt{-a}b\sqrt{b}+81b^2$.

20. Cube $(-1)^{\frac{1}{2}}-(-1)^{\frac{1}{3}}$. *Ans.* $4\sqrt{-1}-2$.

PROBLEM.

(180.) To square a polynomial.

RULE.

Square each term, and annex twice its product into each of the following terms.

PROBLEM.

(181.) Square $2a^2-3b+4c-5d$.

SOLUTION.

Squaring $2a^2$, we have $4a^4$, and annexing twice the product of $2a^2$ into each of the following terms $-3b$, $+4c$, and $-5d$, we obtain $4a^4-12a^2b+16a^2c-20a^2d$. Also squaring $-3b$, we have $9b^2$, and annexing twice the product of $-3b$ into each of the following terms, $+4c$ and $-5d$, we obtain $9b^2-24bc+30bd$. Also squaring $+4c$, we have $16c^2$, and annexing twice the product of $+4c$ into the following term, $-5d$, we obtain $16c^2-40cd$. Also squaring $-5d$, we have $25d^2$, and as there are no terms following $-5d$, we have nothing to annex. Collecting all these results, we have $(2a^2-3b+4c-5d)^2=4a^4-12a^2b+16a^2c-20a^2d+9b^2-24bc+30bd+16c^2-40cd+25d^2$.

EXAMPLES.

1. Square $x+y+z$. *Ans.* $x^2+2xy+2xz+y^2+2yz+z^2$.

2. Square $a-b-c+d$.

Ans. $a^2-2ab-2ac+2ad+b^2+2bc-2bd+c^2-2cd+d^2$.

3. Square $2a^{-\frac{1}{2}}-3a^{\frac{1}{2}}b^{\frac{4}{3}}+a^{\frac{3}{2}}b^{-\frac{1}{3}}$.

Ans. $\frac{4}{a}-12b^{\frac{4}{3}}+4ab^{-\frac{1}{3}}+9ab^{\frac{8}{3}}-6a^2b+a^3b^{-\frac{2}{3}}$.

4. Square $\frac{2b^{-\frac{4}{2}}}{3a^{-\frac{4}{2}}}-\frac{3b^2}{2a^2}+\frac{c^{-\frac{3}{2}}}{d^{-\frac{3}{2}}}$.

Ans. $\frac{4a^4}{9b^4}-2+\frac{4a^2d^{\frac{3}{2}}}{3b^2c^{\frac{3}{2}}}+\frac{9b^4}{4a^4}-\frac{3b^2d^{\frac{3}{2}}}{a^2c^{\frac{3}{2}}}+\frac{d^3}{c^3}$.

PROBLEM.

(182.) To cube a polynomial.

RULE.

Cube each term, and annex to it three times the product of its square into each of the other terms, and also SIX *times all the products that can be formed by multiplying three different terms together.*

PROBLEM.

(183.) Cube $a+b+c+d$.

SOLUTION.

Cubing a and multiplying three times its square into the other terms, gives $a^3+3a^2b+3a^2c+3a^2d$.

Cubing b and multiplying three times its square into the other terms, gives $b^3+3b^2a+3b^2c+3b^2d$.

Cubing c and multiplying three times its square into the other terms, gives $c^3+3c^2a+3c^2b+3c^2d$.

Cubing d and multiplying three times its square into the other terms, gives $d^3+3d^2a+3d^2b+3d^2c$.

Multiplying by 6 all the products that can be formed of the terms a, $+b$, $+c$, and $+d$, taking three at a time, gives

$6abc+6abd+6acd+6bcd$.

Writing all these results in order, we have $(a+b+c+d)^3=a^3+3a^2b+3a^2c+3a^2d+b^3+3b^2a+3b^2c+3b^2d+c^3+3c^2a+3c^2b+3c^2d+d^3+3d^2a+3d^2b+3d^2c+6abc+6abd+6acd+6bcd$.

Remark.—To find the number of products that can be formed of any number of terms taken three at a time, add the numbers of the following series to as many terms, less 2, as there are given terms:

$$1+3+6+10+15+21+28+36+\&c.$$

If we have seven terms, the number of products there can be formed, taking three at a time, $=1+3+6+10+15=35$.

This may be illustrated as follows: Supposing the seven terms are a, b, c, d, e, f, and g, and we have

$$\left.\begin{array}{l} abc \\ abd \\ abe \\ abf \\ abg \end{array}\right\} 5 \qquad \left.\begin{array}{l} acd \\ ace \\ acf \\ acg \end{array}\right\} 4 \qquad \left.\begin{array}{l} ade \\ adf \\ adg \end{array}\right\} 3 \qquad \left.\begin{array}{l} aef \\ aeg \end{array}\right\} 2 \qquad afg \} 1$$

$$\left.\begin{array}{l} bcd \\ bce \\ bcf \\ bcg \end{array}\right\} 4 \qquad \left.\begin{array}{l} bde \\ bdf \\ bdg \end{array}\right\} 3 \qquad \left.\begin{array}{l} bef \\ beg \end{array}\right\} 2 \qquad bfg \} 1$$

$$\left.\begin{array}{l} cde \\ cdf \\ cdg \end{array}\right\} 3 \qquad \left.\begin{array}{l} cef \\ ceg \end{array}\right\} 2 \qquad cfg \} 1$$

$$\left.\begin{array}{l} def \\ deg \end{array}\right\} 2 \qquad dfg \} 1$$

$$efg \} 1$$

From these results, we have

$$\left.\begin{array}{l} 5+\ 4+3+2+1 \\ 4+\ 3+2+1 \\ 3+\ 2+1 \\ 2+\ 1 \\ 1 \\ \hline 15+10+6+3+1 \end{array}\right\}, \text{ or } \left\{\begin{array}{r} 1+2+3+\ 4+\ 5 \\ +1+2+\ 3+\ 4 \\ 1+\ 2+\ 3 \\ 1+\ 2 \\ 1 \\ \hline 1+3+6+10+15 \end{array}\right.$$

Whence, the law of the series becomes apparent.

EXAMPLES.

1. Cube $a+b+c$.

Ans. $a^3+3a^2b+3a^2c+b^3+3b^2a+3b^2c+c^3+3c^2a+3c^2b+6aoc$.

2. Cube $a+b-c-d$.

Ans. $a^3+3a^2b-3a^2c-3a^2d+b^3+3b^2a-3b^2c-3b^2d-c^3+3c^2a+3c^2b-3c^2d-d^3+3d^2d+3d^2b-3d^2c-6abc-6abd+6acd+6bcd$.

3. Cube $a+2b-c$.

Ans. $a^3+6a^2b-3a^2c+8b^3+12b^2a-12b^2c-c^3+3c^2a+6c^2b-12abc$.

4. Cube $2x^2+4ax-3a^2$.

Ans. $8x^6+48ax^5+60a^2x^4-80a^3x^3-90a^4x^2+105a^5x-27a^6$.

5. Cube $1+x+x^2+x^3$.

Ans. $1+3x+6x^2+10x^3+12x^4+12x^5+10x^6+6x^7+3x^8+x^9$.

PROBLEM.

(184.) To raise a polynomial to any power.

RULE.

Change the polynomial to a binomial, and then proceed according to the principles of the binomial theorem.

PROBLEM.

(185.) Find the 6th power of $a+b+c+d+e$.

SOLUTION.

Changing $a+b+c+d+e$ to a binomial, we have $(a+b+c)+(d+e)$; whence, we have $[(a+b+c)+(d+e)]^6=(a+b+c)^6+6(a+b+c)^5(d+e)+15(a+b+c)^4(d+e)^2+20(a+b+c)^3(d+e)^3+15(a+b+c)^2(d+e)^4+6(a+b+c)(d+e)^5+(d+e)^6$.

In the same way, we find

$$(a+b+c)^6=(a+b)^6+6(a+b)^5c+15(a+b)^4c^2+20(a+b)^3c^3+15(a+b)^2c^4+6(a+b)c^5+c^6.$$

$$(a+b+c)^5=(a+b)^5+5(a+b)^4c+10(a+b)^3c^2+10(a+b)^2c^3+5(a+b)c^4+c^5.$$

$$(a+b+c)^4=(a+b)^4+4(a+b)^3c+6(a+b)^2c^2+4(a+b)c^3+c^4.$$

$$(a+b+c)^3=(a+b)^3+3(a+b)^2c+3(a+b)c^2+c^3.$$

$$(a+b+c)^2=(a+b)^2+2(a+b)c+c^2.$$

Inserting these values in the above expression, we obtain
$(a+b+c+d+e)^6=(a+b)^6+6(a+b)^5c+15(a+b)^4c^2+20(a+b)^3c^3+$
$15(a+b)^2c^4+6(a+b)c^5+c^6+6[(a+b)^5+5(a+b)^4c+10(a+b)^3c^2+$
$10(a+b)^2c^3+5(a+b)c^4+c^5](d+e)+15[(a+b)^4+4(a+b)^3c+$
$6(a+b)^2c^2+4(a+b)c^3+c^4](d+e)^2+20[(a+b)^3+3(a+b)^2c+$
$3(a+b)^2c+c^3](d+e)^3+15[(a+b)^2+2(a+b)c+c^2](d+e)^4+$
$6(a+b+c)(d+e)^5+(d+e)^6$.

EXAMPLES.

1. Find the 4th power of $a+b+c$.
Ans. $a^4+4a^3b+4a^3c+6a^2b^2+12a^2bc+6a^2c^2+4ab^3+12ab^2c+12abc^2$
$+4ac^3+b^4+4b^3c+6b^2c^2+4bc^3+c^4$.

2. Find the 5th power of $a+b+c$.
Ans. $a^5+5a^4b+5a^4c+10a^3b^2+20a^3bc+10a^3c^2+10a^2b^3+30a^2b^2c$
$+30a^2bc^2+10a^2c^3+5ab^4+20ab^3c+30ab^2c^2+20abc^3+5ac^4+b^5$
$+5b^4c+10b^3c^2+10b^2c^3+5bc^4+c^5$.

3. Find the 6th power of $a+b+c$.
Ans. $a^6+6a^5b+6a^5c+15a^4b^2+30a^4bc+15a^4c^2+20a^3b^3+60a^3b^2c$
$+60a^3bc^2+20a^3c^3+15a^2b^4+60a^2b^3c+90a^2b^2c^2+60a^2bc^3+15a^2c^4$
$+6ab^5+30ab^4c+60ab^3c^2+60ab^2c^3+30abc^4+6ac^5+b^6+6b^5c$
$+15b^4c^2+20b^3c^3+15b^2c^4+6bc^5+c^6$.

4. Find the 7th power of $a+b+c$.
Ans. $a^7+7a^6b+7a^6c+21a^5b^2+42a^5bc+21a^5c^2+35a^4b^3+105a^4b^2c$
$+105a^4bc^2+35a^4c^3+35a^3b^4+140a^3b^3c+210a^3b^2c^2+140a^3bc^3$
$+35a^3c^4+21a^2b^5+105a^2b^4c+210a^2b^3c^2+210a^2b^2c^3+105a^2bc^4$
$+21a^2c^5+7ab^6+42ab^5c+105ab^4c^2+140ab^3c^3+105ab^2c^4+42abc^5$
$+7ac^6+b^7+7b^6c+21b^5c^2+35b^4c^3+35b^3c^4+21b^2c^5+7bc^6+c^7$.

5. Find the 4th power of $a+b+c+d$.

6. Find the 5th power of $a+b+c+d$.

REMARK.—The answer to the 5th example contains thirty-five terms and to the 6th fifty-six terms.

CHAPTER VIII.

EVOLUTION.

(186.) Evolution is the reverse of Involution, or is finding a quantity which taken a certain number of times as a factor will produce a given quantity.

PROBLEM.

(187.) To extract a given root of a given monomial.

RULE.

Divide the exponents of the factors of which the monomial is composed by the index of the root, and prefix the sign of the monomial when the index of the root is odd; and when the monomial is positive, and the index of the root is even, prefix + or − written ±.

DEMONSTRATION.

Let $27ab^3c^6$ be a monomial whose cube root is sought. If we resolve $27ab^3c^6$ into three equal factors, one of these factors will be its cube root.

Since $27=3\cdot3\cdot3$, and $a^1=a^{\frac{1}{3}}\cdot a^{\frac{1}{3}}\cdot a^{\frac{1}{3}}$, and $b^3=b\cdot b\cdot b$, and $c^6=c^2\cdot c^2\cdot c^2$, we have $27ab^3c^6=3\cdot3\cdot3\cdot a^{\frac{1}{3}}\cdot a^{\frac{1}{3}}\cdot a^{\frac{1}{3}}\cdot b\cdot b\cdot b\cdot c^2\cdot c^2\cdot c^2=3a^{\frac{1}{3}}bc^2\cdot 3a^{\frac{1}{3}}bc^2\cdot 3a^{\frac{1}{3}}bc^2$; whence, we see that $3a^{\frac{1}{3}}bc^2$ is the cube root of $27ab^3c^6$.

Proceeding according to the rule, we shall arrive at the same result, for since $27ab^3c^6=3^3a^1b^3c^6$, if we divide each of the exponents by 3, we have $3^1a^{\frac{1}{3}}b^1c^2=3a^{\frac{1}{3}}bc^2$.

If the monomial had been $-27ab^3c^6$, we would have had $-27ab^3c^6=-3a^{\frac{1}{3}}bc^2\times-3a^{\frac{1}{3}}bc^2\times-3a^{\frac{1}{3}}bc^2$; whence we see that the cube root of $-27ab^3c^6$ is $-3a^{\frac{1}{3}}bc^2$, or that an odd root has the sign of the monomial.

Also, let a^2 be a monomial whose square root is sought. Since

$a^2=+a.+a$, we see that the square root of a^2 is $+a$. But, since $a^2=-a.-a$, we see that the square root of a^2 may also be $-a$. Whence we derive the fact, that the square root of a positive quantity may be either *plus* or *minus*, which fact is represented by the expression $\sqrt{a^2}=\pm a$.

The double sign $\pm$ should be placed before the square root of a quantity only when we are doubtful in reference to the sign it should have. For if we should be required to square $+a$, and then extract the square root, we know that the only proper result would be $+a$, or the quantity with which we started; also, if we were required to square $-a$ and then extract the square root, we are equally certain the only proper result would be $-a$.

If then, we have a^2, and wish to extract its square root, to give a rigid result, we must know whether this a^2 was obtained by multiplying $+a$ by $+a$, or $-a$ by $-a$; if the former, we know that the square root of this a^2 must be $+a$, and can not be $-a$; but if the latter, we know that the square root of this a^2 must be $-a$, and can not be $+a$. We see then that it is only proper to prefix the double sign $\pm$, when we are ignorant of the signs of the factors which produced the quantity under consideration.

PROBLEM

(188.) 1. Extract the square root of $-a^2$.

SOLUTION.

If we can resolve $-a^2$ into two equal factors, both of which are $+$, or both of which are $-$, then one of these equal factors must be the square root of $-a^2$.

But since neither the product of $+$ by $+$, nor $-$ by $-$ will give $-$, we are certain that $-a^2$ cannot be resolved into two equal positive factors, or into two equal negative factors. Hence, we can not obtain the square root of $-a^2$. The same kind of reasoning will show that we can not obtain the even root of any negative quantity.

The square root of $-a^2$ is represented by $\sqrt{-a^2}$, which expression may be simplified, since $-a^2=a^2.-1$. We can take the square root of a^2, but not of -1; whence, we see that $\sqrt{-a^2}=\sqrt{a^2.-1}=\pm a\sqrt{-1}$.

PROBLEM

2. Extract the square root of a.

SOLUTION.

Following the rule, we have $\pm a^{\frac{1}{2}}$ for the square root of a, that is $\sqrt{a}=\pm a^{\frac{1}{2}}$. But we have gained nothing, since $\pm a^{\frac{1}{2}}$ is merely another mode of representing that the square root of a is to be extracted. Therefore, the formula $\sqrt{a}=\pm a^{\frac{1}{2}}$ tells us nothing more than that *the square root of* a *is equal to the square root of* a.

If we were to assign to a some positive numerical value, its square root might be obtained either exactly or approximately. The case would, however, be different if we sought the square root of $-a$, and should assign to a some numerical value, because, we can never extract the square root of -1, for $\sqrt{-a}$ is equal to $\sqrt{a}\sqrt{-1}$, the value of which could be obtained either exactly or approximately provided the value of $\sqrt{-1}$ could be ascertained. But we have seen that the square root of a negative quantity can not be obtained either exactly or approximately.

PROBLEM

3. Extract the mnth root of a.

SOLUTION.

To find the mnth root of a^1, we must divide the exponent of a by mn, which gives $a^{\frac{1}{mn}}$, that is $\sqrt[mn]{a}=a^{\frac{1}{mn}}$.

Let us now take the nth root of a^1, which is $a^{\frac{1}{n}}$, and take the mth root of this result. To do this, we must divide the exponent $\frac{1}{n}$ by m, which gives $a^{\frac{1}{mn}}$ for the mth root of $a^{\frac{1}{n}}$. Hence, we see that we arrive at the same result by extracting the nth root of a quantity and then extracting the mth root of this nth root, as we do by extracting the mnth root of the quantity, that is, $\sqrt[mn]{a}=\sqrt[m]{\sqrt[n]{a}}$.

The mth root of $a^{\frac{1}{n}}$ may also be represented by $\left(a^{\frac{1}{n}}\right)^{\frac{1}{m}}$; therefore, we have $\sqrt[mn]{a}=\sqrt[m]{\sqrt[n]{a}}=a^{\frac{1}{mn}}=\left(a^{\frac{1}{n}}\right)^{\frac{1}{m}}$, or $\left(a^{\frac{1}{m}}\right)^{\frac{1}{n}}$.

PROBLEM

4. Extract the square root of 32.

SOLUTION.

This can be approximately done by the arithmetical rule. But it

is sometimes advisable to make the square root of a surd depend on some smaller surd.

Since $32 = 16 \cdot 2$, we can easily get the approximate square root of 32 if we already know the approximate square root of 2, by multiplying the approximate square root of 2 by ± 4, the square root of 16; because $\sqrt{32} = \sqrt{16 \cdot 2} = \sqrt{16}\sqrt{2} = \pm 4\sqrt{2}$.

PROBLEM

5. Extract the square root of $(x^3 + 2x^2y + xy^2)$.

SOLUTION.

By factoring, we get $(x^3 + 2x^2y + xy^2) = (x^2 + 2xy + y^2)x$. We know by Theorem 1. **(109)**, that $(x^2 + 2xy + y^2) = (x + y)^2$; hence, we have $(x^3 + 2x^2y + xy^2) = (x + y)^2x$. Extracting the square root of $(x+y)^2x$, we get $\pm(x+y)\sqrt{x}$.

EXAMPLES.

1. Extract the square root of $4a^2b^2$. *Ans.* $\pm 2ab$.

2. Extract the square root of -25. *Ans.* $\pm 5\sqrt{-1}$.

3. Extract the square root of $(a+b)^2$. *Ans.* $\pm(a+b)$.

4. Extract the cube root of $64a^6b^2$. *Ans.* $4a^2\sqrt[3]{b^2}$.

5. Extract the square root of 8. *Ans.* $\pm 2\sqrt{2}$.

6. Extract the square root of 24. *Ans.* $\pm 2\sqrt{6}$.

7. Extract the square root of 68. *Ans.* $\pm 2\sqrt{17}$.

8. Extract the square root of 150. *Ans.* $\pm 5\sqrt{6}$.

9. Extract the fifth root of $-32a^5x^{10}y^{15}$. *Ans.* $-2ax^2y^3$.

10. Extract the cube root of -40. *Ans.* $-2\sqrt[3]{5}$.

11. Extract the square root of $-16a^2b^{\frac{2}{3}}c^3$. *Ans.* $\pm 4ab^{\frac{1}{3}}c\sqrt{-c}$.

12. Extract the square root of $\frac{833}{2057}$. *Ans.* $\pm\frac{7}{11}$.

13. Extract the square root of $4a^6(a^2 + 2ab + b^2)$.
Ans. $\pm 2a^3(a+b)$.

14. Extract the cube root of $-\frac{54}{128}$. *Ans.* $-\frac{3}{4}$.

15. Extract the cube root of $(x^3+y^3+3x^2y+3xy^2)$.

Ans. $(x+y)$.

16. Extract the mth root of $a^{mn}b^{mp}c^{-mq}d^{-mr}$. *Ans.* $\frac{a^nb^p}{c^qd^r}$.

17. Extract the fifth root of -1. *Ans.* -1.

18. Extract the fourth root of $-a^4$. *Ans.* $\pm a\sqrt{-1}$.

19. Find the value of $\sqrt[3]{a^3+a^3b^2}$. *Ans.* $a\sqrt[3]{1+b^2}$.

20. Find the value of $3\sqrt[3]{108}$. *Ans.* $9\sqrt[3]{4}$.

PROBLEM.

(189.) To extract the nth root of a given quantity to within a given fraction.

RULE.

Represent the given quantity in the form of a whole number, and the given fraction with one for its numerator, and then multiply the given quantity by the nth *power of the denominator, and extract the* nth *root of the product to the nearest unit, and divide the result by the denominator of the fraction.*

DEMONSTRATION.

Let $\frac{a}{b}$ be a quantity whose nth root is sought to within $\frac{c}{d}$. Representing $\frac{a}{b}$ in the form of a whole number, we have ab^{-1} which we shall put equal to Q, and representing $\frac{c}{d}$ in the form of a fraction with 1 for its numerator, we have $\frac{1}{\frac{d}{c}}$ which becomes $\frac{1}{m}$ by putting m for $\frac{d}{c}$.

We have then the quantity Q whose nth root is sought to within $\frac{1}{m}$. $Q=\frac{Qm^n}{m^n}$. Let r be the root of the greatest nth power contained in Qm^n; then the value $\frac{Qm^n}{m^n}$ is greater than the value of $\frac{r^n}{m^n}$ and

less than the value of $\dfrac{(r+1)^n}{m^n}$; therefore, the nth root of $\dfrac{Qm^n}{m^n}$ or of Q is greater than $\dfrac{r}{m}$, the nth root of $\dfrac{r^n}{m^n}$, and less than $\dfrac{r+1}{m}$, the nth root of $\dfrac{(r+1)^n}{m^m}$, or in other words, the difference between the nth root of Q and $\dfrac{r}{m}$ is less than the difference between $\dfrac{r}{m}$ and $\dfrac{r+1}{m}$; but the difference between $\dfrac{r}{m}$ and $\dfrac{r+1}{m}$ is $\dfrac{1}{m}$; hence, by taking $\dfrac{r}{m}$ for the nth root of Q, we have a result which differs from the true result by less than $\dfrac{1}{m}$. *Q. E. D.*

PROBLEM.

(190.) Extract the square root of 9 to within $\dfrac{2}{3}$.

SOLUTION.

Since $\dfrac{2}{3}=\dfrac{1}{1\frac{1}{2}}$, we must square $1\frac{1}{2}$ and multiply 9 by the result; whence, we get $9 \cdot 2\frac{1}{4}=20\frac{1}{4}$. We take either 4 or 5 for the square root of $20\frac{1}{4}$ to within a unit. If we take 4 and divide it by $1\frac{1}{2}$, we have $2\frac{2}{3}$ for the square root of 9 to within $\frac{2}{3}$; but if we take 5 and divide it by $1\frac{1}{2}$, we have also $3\frac{1}{3}$ for the square root of 9 to within $\frac{2}{3}$. In these two results, we observe that the first is just $\frac{1}{3}$ less than the true root, and the second $\frac{1}{3}$ greater.

NOTE.—In the following examples we shall sometimes give both results, placing the most accurate first.

EXAMPLES.

1. Extract the square root of 5 to within $\frac{1}{3}$. *Ans.* $2\frac{1}{3}$ or 2.

2. Extract the square root of 8 to within $\frac{1}{9}$. *Ans.* $2\frac{7}{9}$ or $2\frac{8}{9}$.

3. Extract the square root of 7 to within $\frac{1}{7}$. *Ans.* $2\frac{5}{7}$ or $2\frac{4}{7}$.

4. Extract the square root of $\frac{3}{4}$ to within $\frac{1}{2}$. *Ans.* $\frac{1}{2}$.

5. Extract the square root of $\frac{5}{7}$ to within $\frac{1}{7}$. *Ans.* $\frac{5}{7}$.

6. Extract the square root of $\frac{9}{16}$ to within $\frac{1}{4}$. *Ans.* $\frac{3}{4}$.

7. Extract the square root of $\frac{11}{16}$ to within $\frac{1}{16}$. *Ans.* $\frac{13}{16}$.

8. Extract the square root of $\frac{7}{6}$ to within $\frac{1}{5}$. *Ans.* 1.

9. Extract the cube root of 5 to within $\frac{1}{5}$. *Ans.* $1\frac{3}{5}$.

10. Extract the cube root of 10 to within $\frac{1}{7}$. *Ans.* $2\frac{1}{7}$.

11. Extract the fourth root of 7 to within $\frac{1}{2}$. *Ans.* $1\frac{1}{2}$.

12. Extract the square root of $\frac{2}{3}$ to within $\frac{5}{7}$. *Ans.* $\frac{5}{7}$.

PROBLEM.

(191.) To extract the square root of a polynomial.

RULE.

1. *Arrange the given polynomial according to the powers of a certain letter.*

2. *Extract the square root of the first term of the polynomial, and place the result as the first term of the required root.*

3. *Subtract the square of the first term of the root from the given polynomial.*

4. *Divide the first term of the remainder by twice the first term of the root, and place the quotient as the second term of the root.*

5. *Annex the second term of the root to twice the first term of the root, and multiply the sum by the second term of the root, and subtract the product from the first remainder.*

6. *Divide the first term of the second remainder by twice the first term of the root, and place the quotient as the third term of the root.*

7. *Annex the third term of the root to twice the sum of the two preceding terms, and multiply the trinomial thus formed by the third term of the root, and subtract the product from the second remainder. Then proceed in the same manner to find the other terms of the root.*

DEMONSTRATION.

The accuracy of this rule depends upon the following principles.

PRINCIPLE

1. *When a polynomial and its square are arranged according to the powers of a certain letter, the first term of the square is the square of the first term of the polynomial.*

This principle is only a particular case of that upon which the division of polynomials is based.

PRINCIPLE

2. *When a polynomial and its square are arranged according to the powers of a certain letter, and the square of the sum of* n *terms of the polynomial is subtracted from its complete square, the first term of the remainder is twice the product of the first term of the polynomial by its* (n + 1)st *term.*

Let A represent the first n terms of a polynomial and B the remaining terms. Then the whole polynomial may be represented by $A+B$, which, we will suppose, is arranged according to the decreasing power of a certain letter. If we subtract from the square of $A+B$, which is $A^2+2AB+B^2$, the square of the first n terms of $A+B$, which is A^2, there remains $2AB+B^2$.

It is evident that in the remainder $2AB+B^2$, the first term of $2AB$ contains a higher power of the leading letter than any of the other terms, and is, therefore, the first term of the remainder. But the first term of $2AB$ is twice the product of the first term of A, which is the first term of the proposed polynomial by the first term of B, which is the $(n+1)st$ of the same polynomial. Hence, the principle is established.

Let us now proceed to the extraction of the square root of a polynomial, which, as well as its root, we shall conceive to be arranged according to the powers of a certain letter. By the first Principle, we know that the first term of the polynomial is the square of the first term of the root; hence, we shall obtain the first term of the root by extracting the square root of the first term of the polynomial. By the second Principle, if we subtract from the polynomial the square of the first term of the root, the first term of the remainder will be twice the product of the first term of the root by the second term of the root; hence, we shall obtain the second term of the root by dividing the first term of the remainder by twice the first term of the root. Also, by the same Principle, if we subtract from the proposed polynomial the square of the sum of the first two terms of the root, the first term of the remainder will be twice the product of the first term of the root by the third term of the root; hence, we shall obtain the third term of the root by dividing the first term of this remainder by twice the first term of the root.

We may observe that, instead of subtracting from the given polynomial the square of the sum of the first two terms of the root, it will produce the same result to subtract from the first remainder twice the product of the first term of the root by the second term of the root,

plus the square of the second term of the root, since, we have already subtracted the square of the first term of the root. Therefore, if we write the second term of the root after twice the first term of the root, and multiply the binomial thus formed by the second term of the root, and subtract the product from the first remainder, and divide the first term of this new remainder by twice the first term of the root, we shall obtain the third term of the root. Continuing thus, we shall obtain successively all the terms of which the root is composed, and this process is exactly that given in the rule; hence, the accuracy of the rule is rigidly established.

PROBLEM.

(192.) Extract the square root of $a^4+x^4+6a^2x^2-4a^3x-4ax^3$.

Operation.

$$\begin{array}{r|l} a^4-4a^3x+6a^2x^2-4ax^3+x^4 & a^2-2ax+x^2 \\ a^4 & \\ \hline 2a^2-2ax\ )-4a^3x+6a^2x^2-4ax^3+x^4 & \\ -4a^3x+4a^2x^2 & \\ \hline 2a^2-4ax+x^2\ )\ 2a^2x^2-4ax^3+x^4 & \\ 2a^2x^2-4ax^3+x^4 & \\ \hline \end{array}$$

SOLUTION.

1. We arranged the given polynomial according to the decreasing powers of a.

2. We extracted the square of a^4, and placed the result a^2 as the first term of the root.

3. We subtracted the square of a^2, which is a^4, from the given polynomial, and obtained the remainder, $-4a^3x+6a^2x^2-4ax^3+x^4$.

4. We divided $-4a^3x$ by $2a^2$, and placed the result, $-2ax$, as the second term of the root.

5. We annexed $-2ax$ to $2a^2$, thus making $2a^2-2ax$, and multiplied this binomial by $-2ax$, and subtracted the product, $-4a^3x+4a^2x^2$, from $-4a^3x+6a^2x^2-4ax^3+x^4$, and thus obtained the remainder, $2a^2x^2-4ax^3+x^4$.

6. We divided $2a^2x^2$ by $2a^2$, and placed the result x^2 as the third term of the root.

7. We annexed x^2 to twice a^2-2ax, thus making $2a^2-4ax+x^2$, and multiplied this trinomial by x^2, and subtracted the product, $2a^2x^2-4ax^3+x^4$, from $2a^2x^2-4ax^3+x^4$, and obtained no remainder.

SCHOLIUM.—If we had arranged the given polynomial according to the decreasing powers of x, we should have obtained for the root $x^2-2ax+a^2$, which is the same as $a^2-2ax+x^2$, differently arranged. If we had taken, in the above process, $-a^2$ as the root of a^4, the complete result would have been $-a^2+2ax-x^2$, or $2ax-(a^2+x^2)$. The two roots of the polynomial are given by the expression $\pm(a^2-2ax+x^2)$.

EXAMPLES.

1. Extract the square root of $a^2+2ab+b^2$. *Ans.* $\pm(a+b)$.

2. Extract the square root of $a^{2m}-2a^mb^{\frac{n}{2}}+b^n$.

Ans. $a^m-b^{\frac{n}{2}}$ or $b^{\frac{n}{2}}-a^m$.

3. Extract the square root of $25x-70x^{\frac{3}{4}}+49x^{\frac{1}{2}}$.

Ans. $5x^{\frac{1}{2}}-7x^{\frac{1}{4}}$ or $7x^{\frac{1}{4}}-5x^{\frac{1}{2}}$.

4. Extract the square root of $\dfrac{4a^2}{b^2}-\dfrac{12ac}{bd}+\dfrac{9c^2}{d^2}$.

Ans. $\dfrac{2a}{b}-\dfrac{3c}{d}$ or $\dfrac{3c}{d}-\dfrac{2a}{b}$.

5. Extract the square root of $\dfrac{x^3}{16}-\frac{1}{6}x^{\frac{9}{4}}+\frac{1}{9}x^{\frac{3}{2}}$

Ans. $\frac{1}{4}x^{\frac{3}{2}}-\frac{1}{3}x^{\frac{3}{4}}$ or $\frac{1}{3}x^{\frac{3}{4}}-\frac{1}{4}x^{\frac{3}{2}}$.

6. Extract the square root of $9a^nx^{\frac{m}{2}}-42a^{\frac{n+m}{2}}x^{\frac{n+m}{4}}+49a^mx^{\frac{n}{2}}$.

Ans. $3a^{\frac{n}{2}}x^{\frac{m}{4}}-7a^{\frac{m}{2}}x^{\frac{n}{4}}$ or $7a^{\frac{m}{2}}x^{\frac{n}{4}}-3a^{\frac{n}{2}}x^{\frac{m}{4}}$.

7. Extract the square root of $10x^4-10x^3-12x^5+9x^6-2x+1+5x^2$.

Ans. $\pm(3x^3-2x^2+x-1)$.

8. Extract the square root of $\dfrac{2ab^2x^3+b^4x^4+a^2x^2}{2a^mx^n+a^{2m}+x^{2n}}$.

Ans. $\pm\dfrac{ax+b^2x^2}{a^m+x^m}$.

9. Extract the square root of $4x^3-20x^{\frac{9}{4}}a^{\frac{2}{3}}+25x^{\frac{3}{2}}a^{\frac{4}{3}}+24x^{\frac{3}{2}}y^{\frac{5}{6}}b^{\frac{1}{2}}$ $-60x^{\frac{3}{4}}y^{\frac{5}{6}}a^{\frac{2}{3}}b^{\frac{1}{2}}+36by^{\frac{5}{3}}$. *Ans.* $\pm(2x^{\frac{3}{2}}-5x^{\frac{3}{4}}a^{\frac{2}{3}}+6y^{\frac{5}{6}}b^{\frac{1}{2}})$.

10. Extract the square root of $a^4-2a^3+\frac{3}{2}a^2-\frac{1}{2}a+\frac{1}{16}$.

Ans. $\pm(a^2-a+\frac{1}{4})$.

PROBLEM.

(193.) To extract the cube root of a polynomial.

RULE.

1. *Arrange the given polynomial according to the powers of a certain letter.*

2. *Extract the cube root of the first term of the polynomial, and place the result as the first term of the required root.*

3. *Subtract the cube of the first term of the root from the given polynomial.*

4. *Divide the first term of the remainder by three times the square of the first term of the root, and place the quotient as the second term of the root.*

5. *Cube the sum of the first two terms of the root, and subtract the result from the given polynomial.*

6. *Divide the first term of the second remainder by three times the square of the first term of the root, and place the results as the third term of the root.*

7. *Cube the sum of the first three terms of the root, and subtract the result from the given polynomial. Then proceed in the same manner to find the other terms of the root.*

DEMONSTRATION.

The accuracy of this rule depends on the following principles.

PRINCIPLE

1. *When a polynomial and its cube are arranged according to the powers of a certain letter, the first term of the cube is the cube of the first term of the polynomial.*

This principle is only a particular case of that upon which the division of polynomials is based.

PRINCIPLE

2. *When a polynomial and its cube are arranged according to the powers of a certain letter, and the cube of the sum of n terms of the polynomial is subtracted from its complete cube, the first term of the remainder is three times the product of the square of the first term of the polynomial by its* $(n+1)$st *term.*

Let A represent the first n terms of a polynomial, and B the remaining terms. Then the whole polynomial may be represented by $A+B$, which we will suppose is arranged according to the decreasing powers of a certain letter. If we subtract from the cube of $A+B$, which is $A^3+3A^2B+3AB^2+B^3$, the cube of the first n terms of $A+B$, which is A^3, there remains $3A^2B+3AB^2+B^3$. It is evident that in the remainder $3A^2B+3AB^2+B^3$ the first term of $3A^2B$ contains a higher power of the leading letter than any of the other terms, and is therefore the first term of the remainder. But the first term of $3A^2B$ is three times the product of the square of the first term of A, which is the first term of the proposed polynomial, by the first term of B, which is the $(n+1)st$ term of the same polynomial. Hence the principle is established.

Let us proceed now to the extraction of the cube root of a polynomial which, as well as its root, we shall conceive to be arranged according to the powers of a certain letter. By the first Principle, we know that the first term of the polynomial is the cube of the first term of the root; hence, we shall obtain the first term of the root by extracting the cube root of the first term of the polynomial. By the second Principle, if we subtract from the polynomial the cube of the first term of the root, the first term of the remainder will be three times the product of the square of the first term of the root by the second term of the root; hence, we shall obtain the second term of the root by dividing the first term of the remainder by three times the square of the first term of the root. Also, by the same principle, if we subtract from the proposed polynomial the cube of the sum of the first two terms of the root, the first term of the remainder will be three times the product of the square of the first term of the root by the third term of the root; hence, we shall obtain the third term of the root by dividing the first term of this remainder by three times the square of the first term of the root. Continuing thus, we shall obtain all the terms of the root. This process is exactly that given in the rule; hence, the accuracy of the rule is rigidly established.

PROBLEM.

(194.) Extract the cube root of $x^6-6x^5+21x^4-44x^3+63x^2-54x+27$.

Operation.

$$
\begin{array}{l}
x^6-6x^5+21x^4-44x^3+63x^2-54x+27\,(x^2-2x+3.\\
\underline{x^6}\\
3x^4)\;-6x^5\\
\underline{x^6-6x^5+12x^4-\;8x^3}\\
\qquad 3x^4)9x^4\\
\underline{x^6-6x^5+21x^4-44x^3+63x^2-54x+27.}
\end{array}
$$

REMARK.—The student should observe that it is not necessary to bring down any terms of the remainder except the first, as was done in the above operation.

SOLUTION.

1. We arranged the given polynomial according to the decreasing powers of x.

2. We extracted the cube root of x^6 and placed the result, x^2, as the first term of the root.

3. We subtracted the cube of x^2 which is x^6 from the given polynomial.

4. We divided $-6x^5$, the first term of the remainder, by $3x^4$, or three time the square of x^2, and placed the result, $-2x$, as the second term of the root.

5. We cubed x^2-2x, which is the sum of the first two terms of the root, and subtracted the result, $x^6-6x^5+12x^4-8x^3$, from the given polynomial.

6. We divided $9x^4$, the first term of the second remainder, by $3x^4$, or three times the square of x^2, and placed the result, 3, as the third term of the root.

7. We cubed x^2-2x+3 which is the sum of the first three terms of the root and subtracted the result, $x^6-6x^5+21x^4-44x^3+63x^2-54x+27$, from the given polynomial, which left no remainder.

EXAMPLES.

1. Extract the cube root of $a^3+3a^2b+3ab^2+b^3$. *Ans.* $a+b$.

2. Extract the cube root of $8a^3x^3-84a^2bx^4+294ab^2x^5-343b^3x^6$. *Ans.* $2ax-7bx^2$.

3. Extract the cube root of $8x^6-36ax^5+102a^2x^4-171a^3x^3+204a^4x^2-144a^5x+64a^6$. *Ans.* $2x^2-3ax+4a^2$.

4. Extract the cube root of $x^6-9x^5+39x^4-99x^3+156x^2-144x+64$. *Ans.* x^2-3x+4.

5. Extract the cube root of $x^6+6x^5-40x^3+96x-64$.

Ans. x^2+2x-4.

6. Extract the cube root of $x^6-6x^6+15x^4-20x^3+15x^2-6x+1$.

Ans. x^2-2x+1.

7. Extract the cube root of $a^3+3a^2b+3a^2c+3ab^2+3ac^2+6abc+b^3+3b^2c+3bc^2+c^3$.

Ans. $a+b+c$.

8. Extract the cube root of $27x^6-54x^5+63x^4-44x^3+21x^2-6x+1$.

Ans. $3x^2-2x+1$.

9. Extract the cube root of $(a+b)^3+3(a+b)^2c+3(a+b)c^2+c^3$.

Ans. $a+b+c$.

10. Extract the cube root of $1-6x+12x^2-8x^3$. *Ans.* $1-2x$.

PROBLEM.

(195.) To find the mth root of a polynomial.

RULE.

1. *Arrange the polynomial according to the powers of a certain letter.*

2. *Extract the* mth *root of the first term of the polynomial, and place the result as the first term of the root.*

3. *Raise the first term of the root to the* mth *power, and subtract the result from the given polynomial.*

4. *Divide the first term of the remainder by* m *times the* $(m-1)$st *power of the first term of the root, and place the quotient as the second term of the root.*

5. *Raise the sum of the first two terms of the root to the* mth *power, and subtract the result from the given polynomial.*

6. *Divide the first term of the second remainder by* m *times the* $(m-1)$st *power of the first term of the root, and place the quotient as the third term of the root. Continue thus until all the terms of the root are obtained.*

DEMONSTRATION

To be supplied by the student.

PROBLEM.

(196.) Extract the 4th root of $a^4-4a^3x+x^4+6a^2x^2-4ax^3$.

Operation.

$$\begin{array}{l} a^4-4a^3x+6a^2x^2-4ax^3+x^4(a-x \\ a^4 \\ 4a^3)\ -4a^3x \\ \overline{a^4-4a^3x+6a^2x^2-4ax^3+x^4} \end{array}$$

SOLUTION.

1. We arranged the polynomial according to the decreasing powers of a.

2. We extracted the 4th root of a^4 and placed the result, a, as the first term of the root.

3. We raised $a-x$ to the 4th power, and obtained $a^4-4a^3x+6a^2x^2-4ax^3+x^4$, which, subtracted from the given polynomial, left no remainder.

EXAMPLES.

1. Find the second root of $x^2+2xy+y^2+6xz+6yz+9z^2$.
Ans. $\pm(x+y+3z)$.

2. Find the third root of $a^3-6a^2x+12ax^2-8x^3$.
Ans. $a-2x$.

3. Find the fourth root of $16a^4-96a^3x+216a^2x^2-216ax^3+81x^4$.
Ans. $\pm(2a-3x)$.

4. Find the sixth root of $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.
Ans. $\pm(x-1)$.

5. Find the eighth root of $x^8+8x^7+28x^6+56x^5+70x^4+56x^3+28x^2+8x+1$.
Ans. $\pm(x+1)$.

CHAPTER IX.

RADICALS.

(197.) A *radical expression* is one which contains one or more radical signs, or fractional exponents; as, $\sqrt{a}$, $a^{\frac{1}{2}}$, $\sqrt[4]{a^2}$, &c.

(198.) A *rational quantity* is one which may be represented without the aid of a radical expression. Thus, $\sqrt{a^2}$, $8^{\frac{1}{3}}$, and $\sqrt[4]{81}$ are rational quantities, because they may be represented $\pm a$, 2, and ± 3.

(199.) An *irrational* or *surd quantity* is one which can not be represented without the aid of a radical expression; as, $\sqrt{a}$, $\sqrt{5}$, $2^{\frac{1}{3}}$, &c.

(200.) An *imaginary quantity* is one which is represented by a radical expression denoting the even root of a negative quantity; as, $\sqrt{-1}$, $\sqrt[4]{-2}$, $(-4)^{\frac{1}{2}}$, &c.

(201.) A *quadratic surd* is one in which the root indicated is the square root; as, $\sqrt{a}$, $\sqrt{b}$, $\sqrt{2}$, &c.

(202.) A *cubic surd* is one in which the root indicated is the cube root; as, $\sqrt[3]{a}$, $b^{\frac{1}{3}}$, $\sqrt[3]{2}$, &c.

THEOREM.

(203.) *The square root of a quantity can not be partly rational and partly a quadratic surd.*

DEMONSTRATION.

If possible, let the square root of a be $b+\sqrt{c}$. It follows then that a must be equal to the square of $b+\sqrt{c}$, or $b^2+2b\sqrt{c}+c$, that is, a rational quantity is equal to the sum of two rational quantities and an irrational one, which is manifestly impossible, therefore, $\sqrt{a}=b+\sqrt{c}$ represents an impossibility.

THEOREM.

(204.) *In any equation consisting of rational quantities and*

quadratic surds, the sum of the rational quantities on each side of the sign of equality is equal, and also, the sum of the quadratic surds.

DEMONSTRATION.

If $a+b+\sqrt{c}+\sqrt{d}+\sqrt{e}=m+n+p+\sqrt{r}+\sqrt{p}$, then $a+b=m+n+p$, and $\sqrt{c}+\sqrt{d}+\sqrt{e}=\sqrt{r}+\sqrt{p}$. Let $A=a+b$ and $\sqrt{B}=\sqrt{c}+\sqrt{d}+\sqrt{e}$; also, $C=m+n+p$ and $\sqrt{D}=\sqrt{r}+\sqrt{p}$, then $A+\sqrt{B}=C+\sqrt{D}$.

If A does not equal C, let it equal $C\pm t$, then $C\pm t+\sqrt{B}=C+\sqrt{D}$, or $\pm t+\sqrt{B}=\sqrt{D}$, which shows that the square root of D is equal to a quantity partly rational and partly a quadratic surd, which by the last Theorem is impossible, unless $\pm t$ is equal to nothing; whence $\sqrt{B}=\sqrt{D}$ and $A=C$. *Q. E. D.*

THEOREM.

(205.) *If the square root of $a+\sqrt{b}$ equals $x+y$, then the square root of $a-\sqrt{b}$ equals $x-y$; x and y being supposed to be one or both quadratic surds.*

DEMONSTRATION.

By hypothesis $a+\sqrt{b}=x^2+2xy+y^2=x^2+y^2+2xy$. It is evident that x^2+y^2 must be rational, since both x^2 and y^2 are rational, whether x or y are supposed to be rational or quadratic surds. Hence, by the last Theorem, we must have $a=x^2+y^2$ and $\sqrt{b}=2xy$, whence $a-\sqrt{b}=x^2+y^2-2xy=x^2-2xy+y^2$, or $\sqrt{a-\sqrt{b}}=x-y$.

REDUCTION OF SURDS.

PROBLEM.

(206.) To reduce a rational quantity to the form of a proposed surd.

RULE.

Involve the given rational quantity to a power denoted by the index of the proposed surd, and then represent the corresponding root of the result by means of a radical sign or fractional exponent.

PROBLEM.

(207.) Reduce a to the form of a cubic surd.

SOLUTION.

The cube root of a^3 or $\sqrt[3]{a^3}=(a^3)^{\frac{1}{3}}$ is evidently the required form.

EXAMPLES.

1. Reduce 3 to the form of a quadratic surd. *Ans.* $\sqrt{9}$ or $9^{\frac{1}{2}}$.

2. Reduce $3x^2$ to the form of a cubic surd. *Ans.* $\sqrt[3]{27x^6}$.

3. Reduce $\dfrac{abx}{a+b+x}$ to the form of a cubic surd.

Ans. $\sqrt[3]{\dfrac{a^3b^3x^3}{(a+b+x)^3}}$.

4. Reduce $a+x$ to the form of a quadratic surd.

Ans. $\sqrt{a^2+2ax+x^2}$.

5. Reduce $\dfrac{\sqrt{2}}{5}$ to the form of a quadratic surd. *Ans.* $\sqrt{\dfrac{2}{25}}$.

6. Reduce $\dfrac{5}{\sqrt[3]{4}}$ to the form of a cubic surd. *Ans.* $\sqrt[3]{\dfrac{125}{4}}$.

7. Reduce $a^{\frac{1}{5}}b^{\frac{2}{5}}$ to the form of the fifth root. *Ans.* $(ab^2)^{\frac{1}{5}}$.

8. Reduce $\sqrt[3]{a}$ to the form of a quadratic surd. *Ans.* $\sqrt{\sqrt[3]{a^2}}$.

9. Reduce $-a$ to the form of a quadratic surd.

Ans. $\sqrt{a^2}=\sqrt{(-a)^2}$.

10. Reduce $-a$ to the form of the fourth root. *Ans.* $\sqrt[4]{(-a)^4}$.

PROBLEM.

(208.) Reduce $a\sqrt{b}$ to the form of a quadratic surd.

SOLUTION.

Since $a=\sqrt{a^2}$, we have $a\sqrt{b}=\sqrt{a^2}\sqrt{b}=\sqrt{a^2b}$.

EXAMPLES.

1. Reduce $2\sqrt{3}$ to the form of a quadratic surd. *Ans.* $\sqrt{12}$.

2. Reduce $3\sqrt[3]{2}$ to the form of a cubic surd. *Ans.* $\sqrt[3]{54}$.

3. Reduce $a\sqrt[m]{b}$ to the form of the mth root. *Ans.* $\sqrt[m]{a^mb}$.

4. Reduce $(a-b)\sqrt{a^2+b^2+2ab}$ to the form of a quadratic surd.

Ans. $\sqrt{a^4-2a^2b^2+b^4}$.

PROBLEM.

(209.) To reduce two or more radicals having different indices to equivalent ones having the same index.

RULE.

Represent the given radicals by the aid of fractional exponents, and reduce these fractional exponents to equivalent ones having a common denominator; then raise each quantity respectively to the powers denoted by the numerators of these fractions, and the common denominator will be the index of the root of each.

PROBLEM.

(210.) To reduce $2\sqrt[3]{3}$ and $3\sqrt{2}$ to surds expressing the same root.

SOLUTION.

We have $2\sqrt[3]{3}=\sqrt[3]{24}=(24)^{\frac{1}{3}}$ and $3\sqrt{2}=\sqrt{18}=(18)^{\frac{1}{2}}$. But $\frac{1}{3}=\frac{2}{6}$ and $\frac{1}{2}=\frac{3}{6}$; whence, $(24)^{\frac{1}{3}}=(24)^{\frac{2}{6}}=(24^2)^{\frac{1}{6}}=576^{\frac{1}{6}}=\sqrt[6]{576}$, and $(18)^{\frac{1}{2}}=(18)^{\frac{3}{6}}=(18^3)^{\frac{1}{6}}=(5832)^{\frac{1}{6}}=\sqrt[6]{5832}$. Therefore, $2\sqrt[3]{3}=\sqrt[6]{576}$ and $3\sqrt{2}=\sqrt[6]{5832}$.

EXAMPLES.

1. Reduce $\sqrt{2}$ and $\sqrt[3]{4}$ to surds expressing the same root.

Ans. $\sqrt[6]{8}$ and $\sqrt[6]{16}$.

2. Reduce $\sqrt{x^3}$ and $\sqrt[5]{y}$ to surds expressing the same root.

Ans. $\sqrt[10]{x^{15}}$ and $\sqrt[10]{y^2}$.

3. Reduce $\sqrt{ax}$ and $\sqrt[3]{bx^2}$ to surds having a common index.

Ans. $\sqrt[6]{a^3x^3}$ and $\sqrt[6]{b^2x^4}$.

4. Reduce $\sqrt{3}$ and $\sqrt[3]{2}$ to surds having a common index.

Ans. $\sqrt[6]{27}$ and $\sqrt[6]{4}$.

5. Reduce $6^{\frac{2}{3}}$ and $5^{\frac{3}{4}}$ to surds having a common index.

Ans. $\sqrt[12]{6^8}$ and $\sqrt[12]{5^9}$.

6. Reduce $2^{\frac{3}{4}}$, $3^{\frac{2}{3}}$, and $\sqrt{5}$ to surds having a common index.

Ans. $\sqrt[12]{512}$, $\sqrt[12]{6561}$, and $\sqrt[12]{15625}$.

7. Reduce $a\sqrt{a-x}$ and $b\sqrt[3]{a^2-x^2}$ to surds having a common index.

Ans. $\sqrt[6]{a^9-3a^8x+3a^7x^2-a^6x^3}$ and $\sqrt[6]{a^4b^6-2a^2b^6x^2+b^6x^4}$.

8. Reduce $a\sqrt{x-y}$ and $\dfrac{b}{\sqrt[4]{x+y}}$ to surds having a common index.

Ans. $\sqrt[4]{a^4x^2-2a^4xy+a^4y^2}$ and $\sqrt[4]{b^4(x+y)^{-1}}$.

9. Reduce $\sqrt{a^2-x^2}$ and $\sqrt[4]{a^4+x^4}$ to the form of the eighth root.

Ans. $\sqrt[8]{(a^2-x^2)^4}$ and $\sqrt[8]{a^8+2a^4x^4+x^8}$.

10. Reduce $(a+x)^{\frac{1}{2}}$ and $(a-x)^{\frac{2}{3}}$ to surds having a common index.

Ans. $(a^3+3a^2x+3ax^2+x^3)^{\frac{1}{6}}$ and $(a^4-4a^3x+6a^2x^2-4ax^3+x^4)^{\frac{1}{6}}$.

PROBLEM.

(211.) To reduce surds to their simplest form.

RULE.

Separate the quantity under the radical into two factors, one of which must be the greatest perfect power corresponding to the root indicated that is contained in the given quantity. Extract the root of this factor, and place the product of it by the coefficient of the radical part, as the coefficient of the other factor affected by the given radical sign.

PROBLEM

(212.) 1. Reduce $4\sqrt[3]{5(a^3+a^4b)}$ to it simplest form.

SOLUTION.

Since, $\sqrt[3]{5(a^3+a^4b)}=\sqrt[3]{5(1+ab)a^3}=\sqrt[3]{a^3}\,\sqrt[3]{5(1+ab)}=a\sqrt[3]{5(1+ab)}$, we have, $4\sqrt[3]{5(a^3+a^4b)}=4a\sqrt[3]{5(1+ab)}$.

PROBLEM

2. Reduce $\frac{1}{3}\sqrt[3]{\frac{5}{9}}$ to its simplest form.

SOLUTION.

Since, $\sqrt[3]{\frac{5}{9}}=\sqrt[3]{\frac{15}{27}}=\sqrt[3]{\frac{1}{27}\cdot 15}=\sqrt[3]{\frac{1}{27}}\,\sqrt[3]{15}=\frac{1}{3}\sqrt[3]{15}$, we have $\frac{1}{3}\sqrt[3]{\frac{5}{9}}=\frac{1}{9}\sqrt[3]{15}$.

EXAMPLES.

1. Reduce $\sqrt{16a^2x}$ to its simplest form. *Ans.* $4a\sqrt{x}$.

2. Reduce $\sqrt[3]{ab^3x}$ to its simplest form. *Ans.* $b\sqrt[3]{ax}$.

3. Reduce $\sqrt[3]{81}$ to its simplest form. *Ans.* $3\sqrt[3]{3}$.

4. Reduce $\sqrt{288}$ to its simplest form. *Ans.* $12\sqrt{2}$.

5. Reduce $\sqrt{\frac{3}{8}}$ to its simplest form. *Ans.* $\frac{1}{4}\sqrt{6}$.

6. Reduce $\sqrt{27a^3x^5}$ to its simplest form. *Ans.* $3ax^2\sqrt{3ax}$.

7. Reduce $\sqrt[3]{5ax^4-3b^2x^3}$ to its simplest form.

Ans. $x\sqrt[3]{5ax-3b^2}$.

8. Reduce $\sqrt[m]{a^{m+n}b}$ to its simplest form. *Ans.* $a\sqrt[m]{a^nb}$.

9. Reduce $\sqrt{(a+bx)^4xy}$ to its simplest form. *Ans.* $(a+bx)^2\sqrt{xy}$.

10. Reduce $\sqrt[3]{(a+x)^3b^2}$ to its simplest form. *Ans.* $(a+x)b^{\frac{2}{3}}$.

11. Reduce $\sqrt[m]{\frac{(cx-x^2)^{2m}(a+x)}{b+x}}$ to its simplest form.

Ans. $(cx-x^2)^2\left(\frac{a+x}{b+x}\right)^{\frac{1}{m}}$.

12. Reduce $\sqrt[3]{135}$ to its simplest form. *Ans.* $3\sqrt[3]{5}$.

13. Reduce $5\sqrt{54}$ to its simplest form. *Ans.* $15\sqrt{6}$.

14. Reduce $3\sqrt[3]{108}$ to its simplest form. *Ans.* $9\sqrt[3]{4}$.

15. Reduce $\sqrt[3]{ax^3+bx^6}$ to its simplest form. *Ans.* $x\sqrt[3]{a+bx^3}$.

16. Reduce $\frac{1}{2}\sqrt{\frac{3}{7}}$ to its simplest form. *Ans.* $\frac{1}{14}\sqrt{21}$.

17. Reduce $5\sqrt[3]{\frac{2}{3}}$ to its simplest form. *Ans.* $\frac{5}{3}\sqrt[3]{18}$.

18. Reduce $\frac{a}{b}\sqrt{\frac{c^2}{d}}$ to its simplest form. *Ans.* $\frac{ac}{bd}\sqrt{d}$.

19. Reduce $\sqrt{\frac{ab^2}{4(a+x)}}$ to its simplest form.

Ans. $\frac{b}{2(a+x)}\sqrt{a(a+x)}$.

20. Reduce $\sqrt{\frac{ax^2}{a-x}}$ to its simplest form. *Ans.* $\frac{x}{a-x}\sqrt{a(a-x)}$.

ADDITION OF RADICALS.

PROBLEM.

(213.) To add radicals.

RULE.

Reduce the radicals to their simplest form, and proceed as in addition.

PROBLEM

(214.) 1. Add together $\sqrt[3]{500}$ and $\sqrt[3]{108}$.

SOLUTION.

$$\sqrt[3]{500}=\sqrt[3]{125\,.\,4}=\sqrt[3]{125}\sqrt[3]{4}=5\sqrt[3]{4}$$

and $$\sqrt[3]{108}=\sqrt[3]{27\,.\,4}=\sqrt[3]{27}\,\sqrt[3]{4}=3\sqrt[3]{4}$$

Therefore, $$\sqrt[3]{500}+\sqrt[3]{108}=5\sqrt[3]{4}+3\sqrt[3]{4}=8\sqrt[3]{4}.$$

PROBLEM

2. Find the sum of $3\sqrt{\frac{2}{5}}$ and $2\sqrt{\frac{1}{10}}$.

SOLUTION.

$$3\sqrt{\tfrac{2}{5}}=3\sqrt{\tfrac{40}{100}}=3\sqrt{\tfrac{4}{100}\,.\,10}=3\sqrt{\tfrac{4}{100}}\sqrt{10}=3\,.\,\tfrac{2}{10}\sqrt{10}=\tfrac{3}{5}\sqrt{10}$$

and $$2\sqrt{\tfrac{1}{10}}=2\sqrt{\tfrac{10}{100}}=2\sqrt{\tfrac{1}{100}\,.\,10}=2\sqrt{\tfrac{1}{100}}\sqrt{10}=2\,.\,\tfrac{1}{10}\sqrt{10}=\tfrac{1}{5}\sqrt{10}$$

Therefore, $$3\sqrt{\tfrac{2}{5}}+2\sqrt{\tfrac{1}{10}}=\tfrac{3}{5}\sqrt{10}+\tfrac{1}{5}\sqrt{10}=\tfrac{4}{5}\sqrt{10}.$$

EXAMPLES.

1. Find the sum of $\sqrt{18}$ and $\sqrt{8}$. *Ans.* $5\sqrt{2}$.

2. Find the sum of $\sqrt{75}$ and $\sqrt{48}$. *Ans.* $9\sqrt{3}$.

3. Find the sum of $\sqrt{a^2x}$ and $\sqrt{c^2x}$. *Ans.* $(a+c)\sqrt{x}$.

4. Find the sum of $\sqrt{150}$ and $-\sqrt{54}$. *Ans.* $2\sqrt{6}$.

5. Find the sum of $\sqrt[m]{a^mb}$ and $\sqrt[m]{bx^{2m}}$. *Ans.* $(a+x^2)\sqrt[m]{b}$.

6. Find the sum of $\sqrt{4ax^2}$ and $3x\sqrt{9a}$. *Ans.* $11x\sqrt{a}$.

7. Find the sum of $3x\sqrt[3]{2a^5x^2}$, $8a\sqrt[3]{2a^2x^5}$ and $2ax\sqrt[3]{2a^2x^2}$.

Ans. $13ax\sqrt[3]{2a^2x^2}$.

8. Find the sum of $\sqrt{\dfrac{a^4x}{b^3}}$, $\sqrt{\dfrac{a^2x^3}{bc^2}}$, and $\sqrt{\dfrac{a^2c^2x}{bd^2}}$.

Ans. $\left(\dfrac{a^2}{b}+\dfrac{ax}{c}+\dfrac{ac}{d}\right)\sqrt{\dfrac{x}{b}}$.

9. Find the sum of $\sqrt{\dfrac{a^2x-2ax^2+x^3}{a^2+2ax+x^2}}$ and $\sqrt{\dfrac{a^2x+2ax^2+x^3}{a^2-2ax+x^2}}$.

Ans. $2\left(\dfrac{a^2+x^2}{a^2-x^2}\right)\sqrt{x}$.

10. Find the sum of $\sqrt[4]{32}$ and $2\sqrt[3]{40}$. *Ans.* $2\sqrt[4]{2}+4\sqrt[3]{5}$.

11. Find the sum of $\sqrt{24}$, $\sqrt{54}$, and $-\sqrt{6}$. *Ans.* $4\sqrt{6}$.

12. Find the sum of $2\sqrt{8}$, $-7\sqrt{18}$, $5\sqrt{72}$, and $-\sqrt{50}$. *Ans.* $8\sqrt{2}$.

13. Find the sum of $3\sqrt[3]{32}$ and $2\sqrt[3]{54}$. *Ans.* $6(\sqrt[3]{4}+\sqrt[3]{2})$.

14. Find the sum of $\sqrt{24}$, $2\sqrt{72}$, and $a\sqrt{6x^2}$.

Ans. $2(\sqrt{6}+6\sqrt{2})+ax\sqrt{6}$.

15. Find the sum of $8\sqrt{\frac{3}{4}}$, $\sqrt{60}$, $-\frac{11}{5}\sqrt{15}$ and $\sqrt{\frac{3}{5}}$. *Ans.* $4\sqrt{3}$.

16. Find the sum of $\sqrt[3]{81}$, $-2\sqrt[3]{24}$, $\sqrt{28}$ and $2\sqrt{63}$.

Ans. $8\sqrt{7}-\sqrt[3]{3}$.

17. Find the sum of $\sqrt[3]{\dfrac{27a^5x}{2b}}$, and $-\sqrt[3]{\dfrac{a^2x}{2b}}$.

Ans. $(3a-1)\sqrt[3]{\dfrac{a^2x}{2b}}$.

18. Find the sum of $3b^2\sqrt{a^3c}$, $+\dfrac{2}{c}\sqrt{a^5c^3}$ and $-c^4\sqrt{\dfrac{ac}{b^2}}$.

Ans. $\left(3ab^2+2a^2-\dfrac{c^4}{b}\right)\sqrt{ac}$.

19. Find the sum of $\sqrt[3]{54a^{m+6}b^3}$, $-\sqrt[3]{16a^{m-3}b^6}$, $\sqrt[3]{2a^{4m+9}}$, and $\sqrt[3]{2c^3a^m}$.

Ans. $\left(3a^2b-\dfrac{2b^2}{a}+a^{m+3}+c\right)\sqrt[3]{2a^m}$.

20. Find the sum of $\sqrt[m]{2^ma^{mp+3}b^{mn+5}}$, $\sqrt[m]{3^ma^{3m-mn+3}b^{m+5}}$, and $-\sqrt[m]{a^3b^5c^{2m}}$. *Ans.* $(2a^pb^n+3a^{3-n}b-c^2)\sqrt[m]{a^3b^5}$.

PROBLEM.

(215.) Find the sum of $\sqrt{8}$ and $\sqrt{32}$.

SOLUTION.

If we square $\sqrt{8}+\sqrt{32}$ and then take the square root, we would have a quantity equal to the sum of $\sqrt{8}$ and $\sqrt{32}$. The square of $\sqrt{8}+\sqrt{32}$ is $8+2.\sqrt{8}.\sqrt{32}+32$, or $40+2.\sqrt{8.32}$, or $40+2\sqrt{256}$, or $40+2.16$, or $40+32=72$. The square root of 72, or $\sqrt{72}=6\sqrt{2}$, which is the sum of $\sqrt{8}$ and $\sqrt{32}$.

The application of this mode gives exercise in the involution of radicals.

EXAMPLES.

1. Find the sum of $\sqrt{12}$ and $\sqrt{27}$. *Ans.* $5\sqrt{3}$.

2. Find the sum of $\sqrt[3]{16}$ and $\sqrt[3]{54}$. *Ans.* $5\sqrt[3]{2}$.

SUBTRACTION OF RADICALS.

PROBLEM.

(216.) To subtract radicals.

RULE.

Change the sign of the radical to be subtracted and proceed as in addition of radicals.

PROBLEM.

(217.) From $\sqrt{\frac{8}{27}}$ subtract $\sqrt{\frac{1}{6}}$.

SOLUTION.

$$\sqrt{\tfrac{8}{27}}=\sqrt{\tfrac{24}{81}}=\sqrt{\tfrac{4}{81}.6}=\tfrac{2}{9}\sqrt{6}=\tfrac{4}{18}\sqrt{6}$$

and $$\sqrt{\tfrac{1}{6}}=\sqrt{\tfrac{6}{36}}=\sqrt{\tfrac{1}{36}.6}=\tfrac{1}{6}\sqrt{6}=\tfrac{3}{18}\sqrt{6}$$

Therefore, $$\sqrt{\tfrac{8}{27}}-\sqrt{\tfrac{1}{6}}=\tfrac{4}{18}\sqrt{6}-\tfrac{3}{18}\sqrt{6}=\tfrac{1}{18}\sqrt{6}.$$

EXAMPLES.

1. From $\sqrt{108ax^2}$ subtract $\sqrt{48ax^2}$. *Ans.* $2x\sqrt{3a}$.

2. From $9a\sqrt{bx^2}$ take $5x\sqrt{a^2b}$. *Ans.* $4ax\sqrt{b}$.

3. From $\sqrt[m]{a^mb}$ take $\sqrt[m]{bx^{2m}}$. *Ans.* $(a-x^2)\sqrt[m]{b}$.

4. Subtract $a\sqrt{bc^2}$ from $\sqrt[4]{16a^4b^2c^4}$. *Ans.* $ac\sqrt{b}$.

5. Subtract $3\sqrt{45}$ from $5\sqrt{20}$. *Ans.* $\sqrt{5}$.

6. From $\sqrt[3]{192}$ subtract $\sqrt[3]{24}$. *Ans.* $2\sqrt[3]{3}$.

7. From $3\sqrt{\frac{2}{5}}$ subtract $2\sqrt{\frac{1}{10}}$. *Ans.* $\frac{2}{5}\sqrt{10}$.

8. From $\frac{3}{4}\sqrt{\frac{2}{3}}$ subtract $\frac{2}{5}\sqrt{\frac{1}{6}}$. *Ans.* $\frac{11}{60}\sqrt{6}$.

9. From $\sqrt[3]{\dfrac{27a^4x^4}{2b}}$ subtract $\sqrt[3]{\dfrac{ax^4}{54b}}$. *Ans.* $\left(3ax-\dfrac{x}{3}\right)\sqrt[3]{\dfrac{ax}{2b}}$.

10. From $\sqrt{\dfrac{a^2b+2ab^2+b^3}{a^2-2ab+b^2}}$ subtract $\sqrt{\dfrac{a^2b-2ab^2+b^3}{a^2+2ab+b^2}}$.

Ans. $\dfrac{4ab\sqrt{b}}{a^2-b^2}$.

11. From $\sqrt[3]{56}$ subtract $-\sqrt[3]{189}$. *Ans.* $5\sqrt[3]{7}$.

12. From $3\sqrt{a^2b}$ subtract $-3\sqrt{16a^4b}$. *Ans.* $(12a^2+3a)\sqrt{b}$.

MULTIPLICATION OF RADICALS.

PROBLEM.

(218.) To multiply by radicals.

RULE.

Reduce the radicals to equivalent ones expressing the same root, and multiply the coefficients together for the coefficient of the product, and the parts under the radicals for the radical part.

PROBLEM.

(219.) Multiply $\sqrt[m]{a}$ by $\sqrt[n]{b}$.

SOLUTION.

$\sqrt[m]{a}=\sqrt[mn]{a^n}$ and $\sqrt[n]{b}=\sqrt[mn]{b^m}$, whence $\sqrt[m]{a}\cdot\sqrt[n]{b}=\sqrt[mn]{a^n}\cdot\sqrt[mn]{b^m}=\sqrt[mn]{a^nb^m}$.

EXAMPLES.

1. Multiply $\sqrt{2}$ by $\sqrt{5}$. *Ans.* $\sqrt{10}$.

2. Multiply $\sqrt{2}$ by $\sqrt[3]{2}$. *Ans.* $\sqrt[6]{32}$.

3. Multiply $\sqrt{2}$ by $\sqrt[3]{4}$. *Ans.* $\sqrt[6]{128}$.

4. Multiply $5\sqrt{6}$ by $2\sqrt{3}$. *Ans.* $30\sqrt{2}$.

5. Multiply $\sqrt{8}$ by $\sqrt[3]{16}$. *Ans.* $4\sqrt[3]{32}$.

6. Multiply $2\sqrt[3]{\frac{2}{3}}$ by $3\sqrt[3]{\frac{5}{6}}$. *Ans.* $2\sqrt[3]{15}$.

7. Multiply $4\sqrt{3}$ by $3\sqrt[3]{4}$. *Ans.* $12\sqrt[6]{432}$.

8. Multiply $2\sqrt{27}$ by $\sqrt{3}$. *Ans.* 18.

9. Multiply $5a^{\frac{1}{2}}$ by $3a^{\frac{1}{3}}$. *Ans.* $15\sqrt[6]{a^5}$.

10. Multiply together $\sqrt[m]{a}$, $\sqrt[m]{b}$ and $\sqrt[m]{c}$. *Ans.* $\sqrt[m]{abc}$.

11. Multiply together $\sqrt[3]{4}$, $7\sqrt[3]{6}$ and $\frac{1}{2}\sqrt[3]{5}$. *Ans.* $\frac{7}{2}\sqrt[3]{120}$.

12. Multiply together 4, $2\sqrt[6]{3}$ and $\sqrt[6]{72}$. *Ans.* $8\sqrt{6}$.

13. Multiply $\sqrt{a^3x}$ by $\sqrt{ax(a^2-x^2)}$. *Ans.* $a^2x\sqrt{a^2-x^2}$.

14. Multiply together $\dfrac{ax}{bc}\sqrt{ax}$, $\dfrac{by}{cd}\sqrt[3]{by}$ and $\dfrac{c^2d}{a}\sqrt[4]{cz}$.

Ans. $xy\sqrt[12]{a^6b^4c^3x^6y^4z^3}$.

15. Multiply $ab+c\sqrt{xy}$ by $a-\sqrt{z}$.

Ans. $a^2b+ac\sqrt{xy}-ab\sqrt{z}-c\sqrt{xyz}$.

16. Multiply $x-\sqrt{xy}+y$ by $\sqrt{x}+\sqrt{y}$. *Ans.* $x\sqrt{x}+y\sqrt{y}$.

17. Multiply $\sqrt{a^5}+a^2\sqrt[3]{b}+\sqrt{a^3}\sqrt[3]{b^2}+ab+\sqrt{a}\sqrt[3]{b^4}+\sqrt[3]{b^5}$ by $\sqrt{a}-\sqrt[3]{b}$. *Ans.* a^3-b^2.

18. Multiply $\sqrt{2}+\sqrt{3}$ by $2\sqrt{2}-\sqrt{3}$. *Ans.* $1+\sqrt{6}$.

19. Multiply $2\sqrt{6}-3\sqrt{5}$ by $4\sqrt{3}-\sqrt{10}$.

Ans. $39\sqrt{2}-16\sqrt{15}$.

20. Multiply $5+\sqrt[3]{4}+2\sqrt[4]{5}$ by $\sqrt{6}+\sqrt{5}$.

Ans. $5\sqrt{6}+5\sqrt{5}+2\sqrt[4]{125}+2\sqrt[4]{180}+2\sqrt[6]{54}+\sqrt[6]{2000}$.

DIVISION OF RADICALS.

PROBLEM.

(220.) To divide one radical by another.

RULE.

Reduce the radicals to equivalent ones expressing the same root, and divide the coefficient of the dividend by the coefficient of the divisor for the coefficient of the quotient, and the radical part of the dividend by the radical part of the divisor for the radical part of the quotient.

PROBLEM.

(221.) Divide $\frac{1}{2}\sqrt{5}$ by $\frac{2}{3}\sqrt[3]{2}$.

SOLUTION.

$\frac{1}{2}\sqrt{5}=\frac{1}{2}\sqrt[6]{125}$, and $\frac{2}{3}\sqrt[3]{2}=\frac{2}{3}\sqrt[6]{4}$. Dividing $\frac{1}{2}$ by $\frac{2}{3}$, we obtain $\frac{3}{4}$, and $\sqrt[6]{125}$ by $\sqrt[6]{4}$, we get $\sqrt[6]{\frac{125}{4}}$. Therefore, $\frac{1}{2}\sqrt{5}\div\frac{2}{3}\sqrt[3]{2}=\frac{3}{4}\sqrt[6]{\frac{125}{4}}$ $=\frac{3}{4}\sqrt[6]{\frac{125\cdot 16}{64}}=\frac{3}{8}\sqrt[6]{2000}$.

EXAMPLES.

1. Divide $\frac{1}{2}\sqrt{5}$ by $\frac{2}{3}\sqrt{2}$. *Ans.* $\frac{3}{8}\sqrt{10}$.

2. Divide $4\sqrt[6]{32}$ by $\sqrt[3]{16}$. *Ans.* $4\sqrt{\frac{1}{2}}$, or $2\sqrt{2}$.

3. Divide $6\sqrt{12}$ by $\sqrt[3]{24}$. *Ans.* $6\sqrt[6]{3}$.

4. Divide $4\sqrt[3]{ax}$ by $3\sqrt{xy}$. *Ans.* $\dfrac{4}{3xy}\sqrt[6]{a^2x^5y^3}$.

5. Divide $\sqrt{ax^3}$ by $\sqrt{bx}$. *Ans.* $\dfrac{x}{b}\sqrt{ab}$.

6. Divide $\sqrt{ax-x^2}$ by $\sqrt[3]{a^2-x^2}$. *Ans.* $\dfrac{\sqrt{x}\sqrt[6]{a-x}}{\sqrt[3]{a+x)}}$.

7. Divide $a\sqrt{\dfrac{bc-bx}{c}}$ by $b\sqrt{\dfrac{ad-ax}{d}}$. *Ans.* $\dfrac{b}{a}\sqrt{\dfrac{ad(c-x)}{bc(d-x)}}$.

8. Divide $a\sqrt{x}-\sqrt{bx}+a\sqrt{y}-\sqrt{by}$ by $\sqrt{x}+\sqrt{y}$. *Ans.* $a-\sqrt{b}$.

9. Divide $a+b-c+2\sqrt{ab}$ by $\sqrt{a}+\sqrt{b}-\sqrt{c}$. *Ans.* $\sqrt{a}+\sqrt{b}+\sqrt{c}$.

10. Divide $3\sqrt{15}-\sqrt{20}+\sqrt{10}-7$ by $2\sqrt{5}$.

Ans. $\frac{3}{2}\sqrt{3}-1+\frac{1}{2}\sqrt{2}-\frac{1}{10}\sqrt{245}$.

11. Divide $\sqrt{8}+\sqrt[3]{12}+\sqrt[4]{2}$ by $2\sqrt{2}$. *Ans.* $1+\frac{1}{2}\sqrt[6]{18}+\frac{1}{4}\sqrt[4]{8}$.

12. Divide $\sqrt{a^2-x^2}$ by $a-x$. *Ans.* $\sqrt{\frac{a+x}{a-x}}$.

13. Divide $\sqrt{ab^2-b^2c}$ by $\sqrt{a-c}$. *Ans.* b.

14. Divide $\frac{1}{2}\sqrt{\frac{1}{2}}$ by $\sqrt{2}+3\sqrt{\frac{1}{2}}$. *Ans.* $\frac{1}{10}$.

15. Divide $\sqrt{72}+\sqrt{32}-4$ by $\sqrt{8}$. *Ans.* $5-\sqrt{2}$.

16. Divide $4\sqrt[3]{12}$ by $2\sqrt{3}$. *Ans.* $2\sqrt[6]{\frac{16}{3}}$.

17. Divide $\sqrt[4]{\frac{a}{b}}$ by $\sqrt{\frac{a}{b}}$. *Ans.* $\sqrt[4]{\frac{b}{a}}$.

18. Divide $\sqrt[5]{64}$ by 2. *Ans.* $\sqrt[5]{2}$.

19. Divide a by $\sqrt{a}$. *Ans.* $\sqrt{a}$.

20. Divide $\sqrt[5]{ab^{n-1}c^2}$ by $\sqrt[5]{\frac{a^3b^2}{dc^{n-1}}}$. *Ans.* $\sqrt[5]{\frac{b^{n-3}c^{n+1}d}{a^2}}$.

INVOLUTION OF RADICALS.

PROBLEM.

(222.) To raise a radical expression to a given power.

RULE.

Proceed according to the method given in Involution, observing the rules just given in reference to radicals.

PROBLEM.

(223.) 1. Square $3\sqrt[3]{3}$.

SOLUTION.

It is evident that the square of any quantity is equal to the product of the square of its factors. If, then, we multiply the square of $\sqrt[3]{3}$ by 9, we must have the desired result. We know from the nature of fractional exponents, that the square of the cube root of a quantity is equal to the cube root of its square; or, in algebraic language, $(\sqrt[3]{a})^2=\sqrt[3]{a^2}$, because $(a^{\frac{1}{3}})^2=a^{\frac{2}{3}}$. Hence, the square of $\sqrt[3]{3}=\sqrt[3]{9}$ and consequently $(3\sqrt[3]{3})^2=9\sqrt[3]{9}$.

PROBLEM.

2. Raise $\sqrt{3}-\sqrt{2}$ to the 4th power.

SOLUTION.

By the Binomial Theorem, we have

$$(\sqrt{3}-\sqrt{2})^4=(\sqrt{3})^4-4(\sqrt{3})^3(\sqrt{2})+6(\sqrt{3})^2(\sqrt{2})^2-4(\sqrt{3})(\sqrt{2})^3+(\sqrt{2})^4.$$

Simplifying this by the rules for radicals, we obtain

$$(\sqrt{3}-\sqrt{2})^4=9-12\sqrt{6}+36-8\sqrt{6}+4=49-20\sqrt{6}.$$

EXAMPLES.

1. Raise $\frac{1}{6}\sqrt{6}$ to the 4th power. *Ans.* $\frac{1}{36}$.

2. Raise $-\sqrt[3]{a^2}$ to the 4th power. *Ans.* $a^2\sqrt[3]{a^2}$.

3. Raise $17\sqrt{21}$ to the 3d power. *Ans.* $103173\sqrt{21}$.

4. Cube $\sqrt{x}+3\sqrt{y}$. *Ans.* $x\sqrt{x}+27y\sqrt{x}+9x\sqrt{y}+27y\sqrt{y}$.

5. Cube $-\sqrt[3]{\sqrt{a}-\sqrt{bc}}$. *Ans.* $\sqrt{bc}-\sqrt{a}$.

6. Square $\sqrt[4]{ax^3}$. *Ans.* $\sqrt{ax^3}$.

7. Cube $\dfrac{\sqrt{a^2-x^2}}{\sqrt{a}\sqrt[3]{a+x}}$. *Ans.* $\sqrt{\dfrac{(a+x)(a-x)^3}{a^3}}$.

8. Raise $\dfrac{ab}{x^2}\sqrt[5]{(c+x)^5}$ to the 4th power.

Ans. $\dfrac{a^4b^4}{x^8}(c+x)^4$.

9. Cube $a\sqrt[3]{b^3-x^3+3x\sqrt[4]{ax^2}}$. *Ans.* $a^3b^3-a^3x^3+3a^3x\sqrt[4]{ax^2}$.

10. Square $\sqrt{5}+\sqrt{3}$. *Ans.* $8+2\sqrt{15}$.

EVOLUTION OF RADICALS.

PROBLEM.

(224.) Extract the mth root of a given monomial radical.

RULE.

Extract the mth *root of its coefficient, and multiply the result by the* mth *root of the radical, which is obtained by multiplying the index of the radical by the index of the root to be extracted, leaving the quantity under the radical sign unchanged; or by extracting the* mth *root of the quantity under the radical sign, leaving the radical sign unchanged.*

PROBLEM.

(225.) To extract the mth root of $b\sqrt[n]{a^r}$.

SOLUTION.

Following the rule, we get $\sqrt[m]{b\sqrt[n]{a^r}}=b^{\frac{1}{m}}\sqrt[mn]{a^r}=b^{\frac{1}{m}}\sqrt[n]{a^{\frac{r}{m}}}$. When $r=m$, we have $\sqrt[m]{b\sqrt[n]{a^m}}=b^{\frac{1}{m}}\sqrt[n]{a}$.

EXAMPLES.

1. Extract the square root of $\sqrt{a}$. *Ans.* $\sqrt[4]{a}$.

2. Extract the cube root of $\sqrt{a^3}$. *Ans.* $\sqrt{a}$.

3. Extract the square root of $\sqrt{m^2n}$. *Ans.* $\sqrt{m\sqrt{n}}$.

4. Extract the square root of $\sqrt{a^4b^2}$. *Ans.* $a\sqrt{b}$.

5. Extract the square root of $81\sqrt{a}$. *Ans.* $9\sqrt[4]{a}$.

6. Extract the square root of $9\sqrt[3]{3}$. *Ans.* $3\sqrt[6]{3}$.

7. Extract the 4th root of $\frac{16}{81}\sqrt[3]{a^2}$. *Ans.* $\frac{2}{3}\sqrt[6]{a}$.

8. Extract the cube root of $(5a^2-3x^2)^{\frac{3}{2}}$. *Ans.* $\sqrt{5a^2-3x^2}$.

9. Extract the cube root of $\frac{1}{8}a^3\sqrt{b}$. *Ans.* $\frac{1}{2}a\sqrt[6]{b}$.

10. Extract the 4th root of $16a^2\sqrt{x}$. *Ans.* $2\sqrt{a}\sqrt[8]{x}$.

11. Extract the 5th root of $32\sqrt[3]{x^5}$. *Ans.* $2\sqrt[3]{x}$.

12. Extract the nth root of $\sqrt[m]{a^n x^2}$. *Ans.* $a^{\frac{1}{m}} \times {}^{\frac{2}{mn}}$.

13. Extract the cube root of $\frac{a}{3}\sqrt{\frac{a}{3}}$. *Ans.* $\frac{1}{3}\sqrt{3a}$.

14. Extract the cube root of $\frac{1}{8}\sqrt{2}$. *Ans.* $\frac{1}{2}\sqrt[6]{2}$.

15. Extract the cube root of $(a+x)\sqrt{a+x}$. *Ans.* $\sqrt{a+x}$.

16. Extract the cube root of $\sqrt[100]{8x^3}$. *Ans.* $\sqrt[100]{2x}$.

17. Extract the cube root of $27\ \sqrt[11]{27a^9}$. *Ans.* $3\ \sqrt[11]{3a^3}$.

18. Extract the 4th root of $81a^{12}\ \sqrt[4]{4x^2}$. *Ans.* $3a^3\ \sqrt[8]{2x}$.

19. Extract the 18th root of $\sqrt[5]{a^{90}b^{12}}$. *Ans.* $a\ \sqrt[15]{b^2}$.

20. Extract the 5th root of $\sqrt{32x^{10}}$. *Ans.* $x\sqrt{2}$.

PROBLEM.

(226.) To extract the square root of a binomial surd, one of whose terms is rational and the other a quadratic surd.

SOLUTION.

Let $\sqrt{x}+\sqrt{y}$ represent the square root of $a+\sqrt{b}$.

Then, $x+2\sqrt{xy}+y$, or $x+y+2\sqrt{xy}=a+\sqrt{b}$; whence, by Theorem, (**204**), we have $x+y=a$. We know, by Theorem (**205**), that if $\sqrt{x}+\sqrt{y}=\sqrt{a+\sqrt{b}}$, that $\sqrt{x}-\sqrt{y}=\sqrt{a-\sqrt{b}}$. It is evident, then, that the product of $\sqrt{x}+\sqrt{y}$ by $\sqrt{x}-\sqrt{y}$, which is $x-y$, must equal the product of $\sqrt{a+\sqrt{b}}$ by $\sqrt{a-\sqrt{b}}$, which is $\sqrt{a^2-b}$. This gives us $x-y=\sqrt{a^2-b}$. If we extract the square root of half the sum of $x+y$ and $x-y$, the result must equal the square root of half the sum of a and $\sqrt{a^2-b}$, that is $\sqrt{x}=\sqrt{\dfrac{a+\sqrt{a^2-b}}{2}}$.

Also, if we extract the square root of half the difference of $x+y$ and $x-y$, the result must equal the square root of half the difference of a and $\sqrt{a^2-b}$, that is, $\sqrt{y}=\sqrt{\dfrac{a-\sqrt{a^2-b}}{2}}$.

Whence, we have the formula,

$$\sqrt{a+\sqrt{b}}=\sqrt{\frac{a+\sqrt{a^2-b}}{2}}+\sqrt{\frac{a-\sqrt{a^2-b}}{2}}.$$

Letting $\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}$, we would have

$$\sqrt{a-\sqrt{b}}=\sqrt{\frac{a+\sqrt{a^2-b}}{2}}-\sqrt{\frac{a-\sqrt{a^2-b}}{2}}.$$

PROBLEM

(227.) 1. Extract the square root of $19+8\sqrt{3}$.

SOLUTION.

In this problem, $a=19$, and $\sqrt{b}=8\sqrt{3}=\sqrt{192}$, or $b=192$. Putting these values in the first of the above formulas, we have

$$\sqrt{19+\sqrt{192}}=\sqrt{\frac{19+\sqrt{361-192}}{2}}+\sqrt{\frac{19-\sqrt{361-192}}{2}}=$$

$$\sqrt{\tfrac{19+13}{2}}+\sqrt{\tfrac{19-13}{2}}=\sqrt{\tfrac{32}{2}}+\sqrt{\tfrac{6}{2}}=\sqrt{16}+\sqrt{3}=4+\sqrt{3}.$$

Instead of using the formula, we might pursue the operation indicated in obtaining the formula.

PROBLEM

2. Extract the square root of $7+4\sqrt{3}$.

SOLUTION.

This may be solved as the last problem. The following method, however, is generally as easily applied:

$7+4\sqrt{3}=4+4\sqrt{3}+3$. Taking the square root of $4+4\sqrt{3}+3$ by the rule given in Evolution, we have for the result $2+\sqrt{3}$.

It is easier to use the fact that *a trinomial is a perfect square when twice the product of the square roots of two of the terms is equal to the remaining term.*

In this example a bare inspection shows that twice the product of the square roots of 4 and 3 is equal to $4\sqrt{3}$.

Hence, $\sqrt{7+4\sqrt{3}}$, or $\sqrt{4+4\sqrt{3}+3}=2+\sqrt{3}$.

PROBLEM

3. Extract the square root of $\sqrt{32}-\sqrt{24}$.

SOLUTION.

It is obvious that the formula can not be applied to $\sqrt{32}-\sqrt{24}$ as it stands, since both terms are surds. It may, however, be made to assume the proper form, for

$$\sqrt{32}-\sqrt{24}=2\sqrt{8}-\sqrt{3}\sqrt{8}=\sqrt{8}(2-\sqrt{3}).$$

If we extract the square root of $2-\sqrt{3}$, which is of the requisite form, we have, by the second formula,

$$\sqrt{2-\sqrt{3}}=\sqrt{\tfrac{3}{2}}-\sqrt{\tfrac{1}{2}}=\sqrt[4]{\tfrac{9}{4}}-\sqrt[4]{\tfrac{1}{4}}.$$

This, multiplied by the square root of $\sqrt{8}$, which is $\sqrt[4]{8}$, gives $\sqrt[4]{18}-\sqrt[4]{2}$.

Therefore, $\sqrt{\sqrt{32}-\sqrt{24}}=\sqrt[4]{18}-\sqrt[4]{2}$.

PROBLEM

4. Extract the square root of $a^2+2x\sqrt{a^2-x^2}$.

SOLUTION.

Since $a^2+2x\sqrt{a^2-x^2}=x^2+2x\sqrt{a^2-x^2}+a^2-x^2$, a bare inspection shows that its square root is $x+\sqrt{a^2-x^2}$.

EXAMPLES.

1. Extract the square root of $6+\sqrt{20}$. *Ans.* $1+\sqrt{5}$.
2. Extract the square root of $8+\sqrt{39}$. *Ans.* $\frac{1}{2}(\sqrt{26}+\sqrt{6})$.
3. Extract the square root of $11+\sqrt{72}$. *Ans.* $3+\sqrt{2}$.
4. Extract the square root of $6-2\sqrt{5}$. *Ans.* $-1+\sqrt{5}$.
5. Extract the square root of $23-8\sqrt{7}$. *Ans.* $4-\sqrt{7}$.
6. Extract the square root of $33+12\sqrt{6}$. *Ans.* $3+2\sqrt{6}$.
7. Extract the square root of $7-2\sqrt{10}$. *Ans.* $\sqrt{5}-\sqrt{2}$.
8. Extract the square root of $42+3\sqrt{174\frac{2}{9}}$. *Ans.* $\sqrt{14}+2\sqrt{7}$.
9. Extract the square root of $10-\sqrt{96}$. *Ans.* $-2+\sqrt{6}$.

10. Extract the square root of $x-2\sqrt{x-1}$. *Ans.* $1-\sqrt{x-1}$.

11. Extract the square root of $28+5\sqrt{12}$. *Ans.* $5+\sqrt{3}$.

12. Extract the square root of $2+2(1-x)\sqrt{1+2x-x^2}$.

Ans. $1-x+\sqrt{1+2x-x^2}$.

13. Extract the square root of $43-15\sqrt{8}$. *Ans.* $5-3\sqrt{2}$.

14. Extract the square root of $5-\sqrt{24}$. *Ans.* $\sqrt{3}-\sqrt{2}$.

15. Extract the square root of $3-2\sqrt{2}$. *Ans.* $1-\sqrt{2}$.

16. Extract the square root of $87-12\sqrt{42}$. *Ans.* $3\sqrt{7}-2\sqrt{6}$.

17. Extract the square root of $\frac{3}{2}+\sqrt{2}$. *Ans.* $1+\frac{1}{2}\sqrt{2}$.

18. Extract the square root of $2+\sqrt{3}$. *Ans.* $\frac{1}{2}\sqrt{6}+\frac{1}{2}\sqrt{2}$.

19. Extract the square root of $\sqrt{27}+2\sqrt{6}$. *Ans.* $\sqrt[4]{12}+\sqrt[4]{3}$.

20. Extract the square root of $\sqrt{32}+6$. *Ans.* $2+\sqrt{2}$.

21. Extract the square root of $3\sqrt{5}+\sqrt{40}$. *Ans.* $\sqrt[4]{5}+\sqrt[4]{20}$.

22. Extract the square root of $3\sqrt{6}+2\sqrt{12}$. *Ans.* $\sqrt[4]{6}+\sqrt[4]{24}$.

23. Extract the square root of $\sqrt{18}-4$. *Ans.* $\sqrt[4]{8}-\sqrt[4]{2}$.

24. Extract the square root of $12-\sqrt{140}$. *Ans.* $\sqrt{7}-\sqrt{5}$.

25. Extract the square root of $2a+2\sqrt{a^2-b^2}$.

Ans. $\sqrt{a+b}+\sqrt{a-b}$.

26. Extract the square root of $ax-2a\sqrt{ax-a^2}$.

Ans. $a-\sqrt{ax-a^2}$.

27. Extract the square root of $\dfrac{a^2}{4}+\dfrac{c}{2}\sqrt{a^2-c^2}$.

Ans. $\dfrac{c}{2}+\frac{1}{2}\sqrt{a^2-c^2}$.

28. Extract the square root of $(x+xy)-2x\sqrt{y}$.

Ans. $(\sqrt{y}-1)\sqrt{x}$.

29. Extract the square root of $\dfrac{3a}{b}+\sqrt{\dfrac{12a^3c^2}{bd^2}-\dfrac{4a^4c^4}{d^4}}$.

Ans. $\dfrac{ac}{d}+\sqrt{\dfrac{3a}{b}-\dfrac{a^2c^2}{d^2}}$.

30. Extract the square root of $\left(b^2-ab+\frac{a^2}{4}\right)+\sqrt{4ab^3-8a^2b^2+a^3b}$.

Ans. $\sqrt{ab}+\sqrt{b^2-2ab+\frac{1}{4}a^2}$.

PROBLEM.

(228.) To extract the cube root of a binomial $A+B$.

RULE.*

"*Separate either term as A, into two such parts that the one of them may be a cubic number, and the other part divisible by 3 without a remainder; then the cube root of the said cubic part will be one term of the root, and the other term will be the square root of the quotient, arising from dividing the aforesaid third part by the first term just found. So if A be divided into r^3+3s then the root is* $r+\sqrt{\frac{s}{r}}$."

PROBLEM.

(229.) Extract the cube root of $10+\sqrt{108}$.

SOLUTION.

Separate 10 into the two parts, 1 and 9 of which the first is a perfect cube, and the other exactly divisible by 3; whence $r=1$ and $s=3$. Therefore $r+\sqrt{\frac{s}{r}}=1+\sqrt{3}$. We can obtain the same result by separating $\sqrt{108}$ into two parts, $\sqrt{108}=\sqrt{27}+3\sqrt{3}$, the first of which is a perfect cube, and the second exactly divisible by 3; whence, $r=\sqrt{3}$ and $s=\sqrt{3}$. Therefore $r+\sqrt{\frac{s}{r}}=\sqrt{3}+\sqrt{\frac{\sqrt{3}}{\sqrt{3}}}=\sqrt{3}+1$.

EXAMPLES.

1. Extract the cube root of $26+15\sqrt{3}$. *Ans.* $2+\sqrt{3}$.

2. Extract the cube root of $9\sqrt{3}-11\sqrt{2}$. *Ans.* $\sqrt{3}-\sqrt{2}$.

3. Extract the cube root of $135\pm78\sqrt{3}$. *Ans.* $3\pm\sqrt{12}$.

4. Extract the cube root of $72+\sqrt{5120}$. *Ans.* $3+\sqrt{5}$.

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* This rule is given by Hutton in his Mathematical Dictionary, and is credited to *Tortalea.*

5. Extract the cube root of $8+4\sqrt{5}$. *Ans.* $\dfrac{1+\sqrt{5}}{\sqrt[3]{2}}$.

6. Extract the cube root of $68-\sqrt{4374}$. *Ans.* $\dfrac{4-\sqrt{6}}{\sqrt[3]{2}}$.

PROBLEM.

(230.) To extract any root (c) of a binomial surd.

RULE.

"*Let the quantity be $A \pm B$, whereof A is the greater part, and c the exponent [index] of the root required. Seek the least number in whose power n^c is divisible by $AA-BB$ [A^2-B^2], the quotient being Q. Compute $\sqrt[c]{\overline{A+B} \times \sqrt{Q}}$ in the nearest integer number, which suppose to be r. Divide $A\sqrt{Q}$ by its greatest rational divisor, and let the quotient be s, and let $\dfrac{r+\frac{n}{r}}{2s}$, in the nearest integer number, be t; so shall the root required be $\dfrac{ts \pm \sqrt{t^2s^2-n}}{\sqrt[2c]{Q}}$ if the c root of $A \pm B$ can be extracted.*"

Remark.—This rule is taken from *Newton's Algebra*, p. 59. It is there given without any demonstration. *Maclaurin* has attempted a demonstration of it in his *Algebra*, p. 120: Mr. Ryan says the rule "fails when $t=\frac{1}{2}$ exactly; in which case instead of taking t the nearest integer value of $\dfrac{r+\frac{n}{r}}{2s}$, it must be taken equal to $\frac{1}{2}$." He says he proposed to the New York Mathematical Club the following: "Required to know if the cube root of $2\sqrt{7}+3\sqrt{3}$, can be found by the *rule* given by Newton, p. 59, Universal Arithmetic, for extracting any root of a binomial surd; and, if not, to show where that rule fails, and what alteration is to be made in it, so as to obtain the root?" Mr. Ryan states that Dr. Adrian "ably investigated the subject, and found the *rule* not only to fail in this, but in a great variety of other examples; and also discovered the rule to be defective."

PROBLEM

(231.) 1. Extract the cube root of $\sqrt{968}+25$.

SOLUTION.

Here $(\sqrt{968})^2-25^2=A^2-B^2=343$. It is evident since the divisors of 343 are 7, 7, 7; that 7^3 is the least number of the form n^3 that is exactly divisible by 343, whence we have $n=7$, and $Q=1$. $(A+B)\sqrt{Q}$, or $\sqrt{968}+25$ is a very little more than 56, of which the nearest cube root is 4, therefore $r=4$. Dividing $\sqrt{968}=22\sqrt{2}$ $=A\sqrt{Q}$ by its greatest rational divisor 22, we obtain $\sqrt{2}=s$, whence $\dfrac{r+\frac{n}{r}}{2s}$ or $\dfrac{5}{2\sqrt{2}}$ in the nearest integer is $2=t$. Then $ts=2\sqrt{2}$, $\sqrt[3]{t^2s^2-n}=1$, and $\sqrt[2c]{Q}=\sqrt[6]{1}=1$. Hence $2\sqrt{2}+1$, or $1+2\sqrt{2}$ is the cube root of $\sqrt{968}+25$, which result will be proved by trial.

PROBLEM

2. Extract the fifth root of $29\sqrt{6}+41\sqrt{3}$.

SOLUTION.

Here $A^2-B^2=3$, and $n=3$; $Q=81$; $r=5$; $s=\sqrt{6}$; $t=1$; $ts=\sqrt{6}$; $\sqrt{r^2s^2-n}=\sqrt{3}$; and $\sqrt[2c]{Q}=\sqrt[10]{81}=\sqrt[5]{9}$. Consequently trial must be made with $\dfrac{\sqrt{6}+\sqrt{3}}{\sqrt[5]{9}}$, or $\dfrac{\sqrt{3}+\sqrt{6}}{\sqrt[5]{9}}$.

EXAMPLES.

1. Extract the cube root of $\sqrt{242}-12\frac{1}{2}$. *Ans.* $\dfrac{2\sqrt{2}-1}{\sqrt[3]{2}}$.

2. Extract the cube root of $11+5\sqrt{7}$. *Ans.* $\dfrac{\sqrt{7}+1}{\sqrt[3]{2}}$.

3. Extract the fourth root of $49849-2895\sqrt{224}$.

Ans. $5\sqrt{7}-3\sqrt{2}$.

4. Extract the cube root of $2\sqrt{7}+3\sqrt{3}$. *Ans.* $\dfrac{\sqrt{7}+\sqrt{3}}{2}$.

PROBLEM.

(232.) To find such a multiplier, or such multipliers as will make any binomial surd rational.

SOLUTION.

All binomial surds, not imaginary, may be represented by $\sqrt[m]{a}\pm\sqrt[n]{b}$.

Let us then seek a general expression for a multiplier which will render $\sqrt[m]{a}\pm\sqrt[n]{b}$ a rational quantity. We know by Theorem 5, (**113**), that $\dfrac{a^{\frac{r}{m}}-b^{\frac{r}{n}}}{a^{\frac{1}{m}}+b^{\frac{1}{n}}}=Q$, an exact quotient when r is an even number.

Q is a multiplier which will make $a^{\frac{1}{m}}+b^{\frac{1}{n}}$ equal to $a^{\frac{r}{m}}-b^{\frac{r}{n}}$. But we wish $a^{\frac{r}{m}}-b^{\frac{r}{n}}$ to be a rational quantity, which it will be when r is a multiple of both m and n. In this case, however, we must take for r a multiple of m and n, that is also an even number.

By Theorem 6, (**114**), we know that $\dfrac{a^{\frac{r}{m}}+b^{\frac{r}{n}}}{a^{\frac{1}{m}}+b^{\frac{1}{n}}}=Q_1$ an exact quotient when r is an odd number. If r is taken, an odd number, such that it is a multiple of both m and n, Q_1 is the multiple necessary to render $a^{\frac{1}{m}}+b^{\frac{1}{m}}$ rational.

We know by Theorem 4, (**112**), that $\dfrac{a^{\frac{r}{m}}-b^{\frac{r}{n}}}{a^{\frac{1}{m}}-b^{\frac{1}{n}}}=Q_2$ an exact quotient, r being any positive integer. If then r be taken, any multiple of both m and n Q_2 is the multiplier necessary to render $a^{\frac{1}{n}}-b^{\frac{1}{m}}$.

In each of these cases it is evident that r may be assumed to be any of the indefinite number of values which can be found to fulfill the required conditions. The least of these values, however, is the one generally used.

PROBLEM

(**233.**) 1. Find a multiplier that will render $6+\sqrt{7}$ rational.

SOLUTION.

A simple inspection shows that the multiplier is $6-\sqrt{7}$, for $(6+\sqrt{7})(6-\sqrt{7})=36-7=29$, a rational quantity.

Again, we have $\dfrac{a^{\frac{r}{m}}-b^{\frac{r}{n}}}{a^{\frac{1}{m}}+b^{\frac{1}{n}}}=\dfrac{6^{\frac{r}{1}}-7^{\frac{r}{2}}}{6^{\frac{1}{1}}+7^{\frac{1}{2}}}=\dfrac{6^{r}-7^{\frac{r}{2}}}{6+7^{\frac{1}{2}}}$, r must be assumed to be some even number which is a multiple of 1 and 2. Making $r=2$, we have $\dfrac{6^2-7}{6+7^{\frac{1}{2}}}=6-7^{\frac{1}{2}}=Q$ as before.

PROBLEM

2. Find a multiplier that will render $2\sqrt{3}-3\sqrt[3]{2}$ rational.

SOLUTION.

Here $2\sqrt{3}-3\sqrt[3]{2}=\sqrt{12}-\sqrt[3]{54}$; hence, we have $\dfrac{a^{\frac{r}{m}}-b^{\frac{r}{n}}}{a^{\frac{1}{m}}-b^{\frac{1}{n}}}=\dfrac{12^{\frac{r}{2}}-54^{\frac{r}{3}}}{12^{\frac{1}{2}}-54^{\frac{1}{3}}}$ $=\dfrac{12^{\frac{6}{2}}-54^{\frac{6}{3}}}{12^{\frac{1}{2}}-54^{\frac{1}{3}}}=\dfrac{12^3-54^2}{12^{\frac{1}{2}}-54^{\frac{1}{3}}}$. Applying the formula given in the demostration of Theorem 4, (**112**), we get $\dfrac{12^3-54^2}{12^{\frac{1}{2}}-54^{\frac{1}{3}}}=12^{\frac{5}{2}}+12^{\frac{4}{2}}54^{\frac{1}{3}}+12^{\frac{3}{2}}54^{\frac{2}{3}}+12^{\frac{2}{2}}54^{\frac{3}{3}}+12^{\frac{1}{2}}54^{\frac{4}{3}}+54^{\frac{5}{3}}=\sqrt{12^5}+144\sqrt[3]{54}+\sqrt{12^3}\sqrt[3]{54^2}+648+\sqrt{12}\sqrt[3]{54^4}+\sqrt[3]{54^5}$, which simplified is $288\sqrt{3}+432\sqrt[3]{2}+216\sqrt{3}\sqrt[3]{4}+648+324\sqrt{3}\sqrt[3]{2}+486\sqrt[3]{4}$.

EXAMPLES.

1. Find a multiplier that will render $\sqrt{5}+\sqrt{3}$ rational.

Ans. $\sqrt{5}-\sqrt{3}$.

2. Find a multiplier that will render $5+\sqrt{3}$ rational.

Ans. $5-\sqrt{3}$.

3. Find a multiplier that will render $\sqrt{2}-\sqrt{x}$ rational.

Ans. $\sqrt{2}+\sqrt{x}$.

4. Find a multiplier that will render $a\sqrt{b}+b\sqrt{a}$ rational.

Ans. $a\sqrt{b}-b\sqrt{a}$.

5. Find a multiplier that will render $1-\sqrt[3]{2a}$ rational.

Ans. $1+\sqrt[3]{2a}+\sqrt[3]{4a^2}$.

6. Find a multiplier that will render $\sqrt[3]{3}-\frac{1}{2}\sqrt[3]{2}$ rational.

Ans. $\sqrt[3]{9}+\frac{1}{2}\sqrt[3]{6}+\frac{1}{4}\sqrt[3]{4}$.

7. Find a multiplier that will render $\sqrt[4]{5}+\sqrt[4]{3}$ rational.

Ans. $(\sqrt[4]{5}-\sqrt[4]{3})(\sqrt{5}+\sqrt{3})$.

8. Find a multiplier that will render $\sqrt[4]{a^3}+\sqrt[4]{b^3}$ rational.

Ans. $\sqrt[4]{a^9}-\sqrt[4]{a^6b^3}+\sqrt[4]{a^3b^6}-\sqrt[4]{b^9}$.

PROBLEM.

(234.) To reduce a fraction whose denominator is a surd quantity to another that shall have a rational denominator.

SOLUTION.

A simple fraction which has a monominal surd for its denominator, may be represented by $\frac{a}{\sqrt[n]{x}}$, in which a may represent any quantity whatever.

Now, if we multiply both terms of $\frac{a}{\sqrt[n]{x}}$ by $\sqrt[n]{x^{n-1}}$, we have $\frac{a\sqrt[n]{x^{n-1}}}{\sqrt[n]{x^n}}=\frac{a\sqrt[n]{x^{n-1}}}{x}$, which is a fraction having a rational denominator, if x is rational. If x is a binomial surd, it must be rendered rational as in the last problem.

PROBLEM

(235.) 1. Reduce $\frac{2}{\sqrt{3}}$ to a fraction having a rational denominator.

SOLUTION.

Multiply both terms by $\sqrt{3}$, and we have $\frac{2\sqrt{3}}{3}$.

PROBLEM

2. Reduce $\frac{1}{2\sqrt{2+\sqrt{3}}}$ to a fraction having a rational denominator.

SOLUTION.

$$\frac{1}{2\sqrt{2+\sqrt{3}}}=\frac{1}{\sqrt{8+\sqrt{48}}}=\frac{\sqrt{8+\sqrt{48}}}{8+\sqrt{48}}=\frac{(8-\sqrt{48})\sqrt{8+\sqrt{48}}}{64-48}=$$
$$\frac{4(2-\sqrt{3})2\sqrt{2+\sqrt{3}}}{16}=\frac{(2-\sqrt{3})\sqrt{2+\sqrt{3}}}{2}=\tfrac{1}{2}\sqrt{2-\sqrt{3}}.$$

EXAMPLES.

1. Reduce $\frac{3}{\frac{1}{2}\sqrt[4]{5}}$ to a fraction having a rational denominator.

Ans. $\frac{6}{5}\sqrt[4]{125}$.

2. Reduce $\dfrac{3}{\sqrt{5}-\sqrt{2}}$ to a fraction having a rational denominator.

Ans. $\sqrt{5}+\sqrt{2}$.

3. Reduce $\dfrac{\sqrt{2}}{3-\sqrt{2}}$ to a fraction having a rational denominator.

Ans. $\dfrac{2+3\sqrt{2}}{7}$.

4. Reduce $\dfrac{3}{\sqrt[3]{5}-\sqrt[3]{2}}$ to a fraction having a rational denominator.

Ans. $\sqrt[3]{25}+\sqrt[3]{10}+\sqrt[3]{4}$.

5. Reduce $\dfrac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}$ to a fraction having a rational denominator.

Ans. $\dfrac{a-\sqrt{a^2-x^2}}{x}$.

6. Reduce $\dfrac{\sqrt{x^2+x+1}+\sqrt{x^2-x-1}}{\sqrt{x^2+x+1}-\sqrt{x^2-x-1}}$ to a fraction having a rational denominator.

Ans. $\dfrac{x^2+\sqrt{x^4-x^2-2x-1}}{x+1}$.

7. Reduce $\dfrac{2}{\sqrt{5}+\sqrt{3}-\sqrt{2}}$ to a fraction having a rational denominator.

Ans. $\dfrac{2\sqrt{3}-3\sqrt{2}+\sqrt{30}}{6}$.

8. Reduce $\dfrac{1}{\sqrt[3]{3}-\sqrt[3]{2}}$ to a fraction having a rational denominator.

Ans. $\sqrt[3]{9}+\sqrt[3]{6}+\sqrt[3]{4}$.

9. Reduce $\dfrac{8}{\sqrt{3}+\sqrt{2}+1}$ to a fraction having a rational denominator.

Ans. $4+2\sqrt{2}-2\sqrt{6}$.

10. Reduce $\dfrac{2}{\sqrt[4]{5}+\sqrt[4]{3}}$ to a fraction having a rational denominator.

Ans. $\sqrt[4]{125}-\sqrt[4]{75}+\sqrt[4]{45}-\sqrt[4]{27}$.

PROBLEM.

(236.) Transform $2-\sqrt{3}$ to a general surd.

SOLUTION.

Squaring $2-\sqrt{3}$, we have $7-4\sqrt{3}$; if now we indicate the ex-

traction of the square root of this quantity, we shall have $2-\sqrt{3}$ expressed in the form of a general surd, that is $2-\sqrt{3}=\sqrt{7-4\sqrt{3}}$.

EXAMPLES.

1. Transform $\sqrt{a}-2\sqrt{x}$ to a universal surd.

Ans. $\sqrt{a+4x-4\sqrt{ax}}$.

2. Transform $3\sqrt[3]{\frac{1}{3}}+\sqrt[3]{72}$ to a general surd. *Ans.* $3\sqrt[3]{9}$.

3. Transform $\sqrt{27}+\sqrt{48}$ to a universal surd. *Ans.* $7\sqrt{3}$.

4. Transform $\sqrt[3]{320}-\sqrt[3]{40}$ to a general surd. *Ans.* $2\sqrt[3]{5}$.

5. Transform $\sqrt{2}+\sqrt{3}$ to a general surd.

Ans. $\sqrt{5+2\sqrt{6}}$.

6. Transform $\sqrt[3]{2}+\sqrt[3]{4}$ to a general surd.

Ans. $\sqrt[3]{6(1+\sqrt[3]{2}+\sqrt[3]{4})}$.

7. Reduce $\sqrt{2}-2\sqrt{6}$ to a general surd. *Ans.* $\sqrt{26-8\sqrt{3}}$.

8. Reduce $4-\sqrt{7}$ to a general surd. *Ans.* $\sqrt{23-8\sqrt{7}}$.

9. Reduce $\sqrt{2}+\sqrt{3}$ to a general surd. *Ans.* $\sqrt{5+2\sqrt{6}}$.

10. Reduce $2\sqrt[3]{3}-3\sqrt[3]{9}$ to a general surd.

Ans. $\sqrt[3]{162\sqrt[3]{9}-108\sqrt[3]{3}-219}$.

IMAGINARY QUANTITIES.

ADDITION OF IMAGINARY QUANTITIES.

PROBLEM.

(237.) To add $\sqrt{-a^2}$ and $\sqrt{-b^2}$ together.

SOLUTION.

Since, $\sqrt{-a^2}=\sqrt{a^2.-1}=a\sqrt{-1}$, and $\sqrt{-b^2}=\sqrt{b^2.-1}=b\sqrt{-1}$, we have $\sqrt{-a^2}+\sqrt{-b^2}=a\sqrt{-1}+b\sqrt{-1}=(a+b)\sqrt{-1}$.

EXAMPLES.

1. Find the sum of $\sqrt{-4}$ and $\sqrt{-9}$. *Ans.* $5\sqrt{-1}$.

2. Find the sum of $2+\sqrt{-1}$ and $3-\sqrt{-64}$. *Ans.* $5-7\sqrt{-1}$.

3. Find the sum of $\sqrt{-8}$ and $\sqrt{-18}$. *Ans.* $5\sqrt{-2}$.

4. Find the sum of $4\sqrt{-27}$ and $2\sqrt{-12}$. *Ans.* $16\sqrt{-3}$.

5. Find the sum of $\sqrt{-6}$ and $\sqrt{-9}$. *Ans.* $(3+\sqrt{6})\sqrt{-1}$.

6. Find the sum of $\sqrt{-5}$ and $\sqrt{-7}$. *Ans.* $(\sqrt{5}+\sqrt{7})\sqrt{-1}$.

7. Find the sum of $\sqrt[4]{-4}$ and $\sqrt[4]{-9}$. *Ans.* $(\sqrt{2}+\sqrt{3})\sqrt[4]{-1}$.

8. Find the sum of $a+\sqrt{-b}$ and $a+\sqrt{-c}$.
Ans. $2a+(\sqrt{b}+\sqrt{c})\sqrt{-1}$.

9. Find the sum of $\sqrt[4]{-a}$ and $2\sqrt[4]{-a}$. *Ans.* $3\sqrt[4]{-a}$.

10. Find the sum of $\sqrt[4]{-1}$ and $\sqrt[4]{-16}$. *Ans.* $3\sqrt[4]{-1}$.

SUBTRACTION OF IMAGINARY QUANTITIES.

PROBLEM.

(238.) From $\sqrt{-a^2}$ subtract $\sqrt{-b^2}$.

SOLUTION.

Since, $\sqrt{-a^2}=a\sqrt{-1}$, and $\sqrt{-b^2}=b\sqrt{-1}$, we have $a\sqrt{-1}$, $-b\sqrt{-1}=(a-b)\sqrt{-1}$.

EXAMPLES.

1. From $\sqrt{-9}$ subtract $\sqrt{-4}$. *Ans.* $\sqrt{-1}$.

2. From $3-\sqrt{-64}$ subtract $2+\sqrt{-1}$. *Ans.* $1-9\sqrt{1}$.

3. From $\sqrt{-18}$ subtract $\sqrt{-8}$. *Ans.* $\sqrt{-2}$.

4. From $4\sqrt{-27}$ subtract $2\sqrt{-12}$. *Ans.* $8\sqrt{-3}$.

5. From $a+\sqrt{-b}$ subtract $a+\sqrt{-c}$. *Ans.* $(\sqrt{b}-\sqrt{c})\sqrt{-1}$.

6. From $\sqrt[4]{-16}$ subtract $\sqrt[4]{-1}$. *Ans.* $\sqrt[4]{-1}$.

MULTIPLICATION OF IMAGINARY QUANTITIES.

THEOREM.

(239.) An imaginary expression of the form $\sqrt{-A}$ can always be reduced to the form $r\sqrt{-1}$.

DEMONSTRATION.

Let r denote the square root of $+A$. It is evident that r can always be obtained either exactly or approximately when A is a positive rational quantity. Then,

$$\sqrt{-A}=\sqrt{+A\,.-1}=\sqrt{A}\sqrt{-1}=r\sqrt{-1}.$$

PROBLEM.

(240.) To ascertain the rule of signs in the multiplication of imaginary quantities.

SOLUTION.

Let $\sqrt{-a}$ be multiplied by $\sqrt{-a}$. Multiplying, as in the case of surds, not imaginary, we have $\sqrt{-a}\times\sqrt{-a}=\sqrt{-a\times -a}=\sqrt{a^2}=-a$. We put the square root $-a$, because the a^2 was obtained in this case by multiplying $-a$ by $-a$, or, what is the same thing, by squaring $-a$. It may also be observed, that $\sqrt{-a}\times\sqrt{-a}=(\sqrt{-a})^2$. We have already shown, in Evolution, that $(\sqrt{-a})^2=-a$, which result agrees with that just given. Let us now ascertain the product of $\sqrt{-a}$ by $\sqrt{-b}$. Since, $\sqrt{-a}=\sqrt{a}\sqrt{-1}$, and $\sqrt{-b}=\sqrt{b}\sqrt{-1}$, we have $\sqrt{-a}\times\sqrt{-b}=\sqrt{a}\sqrt{-1}\times\sqrt{b}\sqrt{-1}=\sqrt{-1}\sqrt{-1}\sqrt{a}\sqrt{b}=-1\sqrt{ab}=-\sqrt{ab}$.

Whence, we see that the same rule applies in multiplying quadratic imaginary surds as in other surds, provided, however, that the result must be affected by a minus sign. The principles already developed will enable the student to multiply any imaginary quantities in which the index of the root is even.

In all these operations great care must be taken that all the steps be rigid.

NOTE.—The importance of carefully scrutinizing all the operations in which imaginary quantities are concerned can not be better set forth than by showing the positions assumed by different distinguished mathematicians:

"The first idea that occurs on the present subject is, that the square of $\sqrt{-3}$, for example, or the product of $\sqrt{-3}$ by $\sqrt{-3}$, is -3; and, in general, that by multiplying $\sqrt{-a}$ by $\sqrt{-a}$, or, by taking the square of $\sqrt{-a}$, we obtain $-a$. * * * * * *

"Moreover, as $\sqrt{a}$ multiplied by $\sqrt{b}$ makes $\sqrt{ab}$, we shall have $\sqrt{6}$ for the value of $\sqrt{-2}$ multiplied by $\sqrt{-3}$; and $\sqrt{4}$, or 2, for the value of the product of $\sqrt{-1}$ by $\sqrt{-4}$. Thus, we see that two imaginary numbers, multiplied together, produce a real, or possible one.

"But, on the contrary, a possible number, multiplied by an impossible number, gives an imaginary product; thus, $\sqrt{-3}$ by $\sqrt{+5}$, gives $\sqrt{-15}$." —*Euler's Al.*, p. 43.

But Emerson makes the product of imaginaries to be imaginary; and for this reason, that "otherwise a real product would be raised from impossible factors, which is absurd. Thus, $\sqrt{-a}\times\sqrt{-b}=\sqrt{-ab}$, and $\sqrt{-a}\times-\sqrt{-b}=-\sqrt{-ab}$, &c. Also, $\sqrt{-a}\times\sqrt{-a}=-a$, and $\sqrt{-a}\times-\sqrt{-a}=+a$, &c."—*Emerson's Al.*, p. 67.

From a dissertation "On the Arithmetic of Impossible Quantities," by Mr. Playfair, in the *Phil. Trans.* for 1778, p. 318, we learn from some operations there performed, that he makes the product of $\sqrt{-1}$ by $\sqrt{-1}$, or the square of $\sqrt{-1}$, to be -1, and, in another place, he makes the product of $\sqrt{-1}$ by $\sqrt{1-z^2}$ to be $\sqrt{-1+z^2}$.

The authors just quoted not only differ from each other, but each one seems to be inconsistent with himself. Thus, Euber says, $\sqrt{-a}\times\sqrt{-a}=-a$, but $\sqrt{-a}\times\sqrt{-b}=\sqrt{ab}$, and Emerson says, $\sqrt{-a}\times\sqrt{-a}=-a$; but $\sqrt{-a}\times\sqrt{-b}=\sqrt{-ab}$. Now, the formula for the product of $\sqrt{-a}$ by $\sqrt{-b}$ ought to be true whatever values may be assigned to a and b. Let, then, $a=b$. Whence, Euler's formula for the product of $\sqrt{-a}$ by $\sqrt{-b}$, gives $+\sqrt{a^2}=+a$, and Emerson's formula gives $+\sqrt{-a^2}=a\sqrt{-1}$. But they both say that $\sqrt{-a}\times\sqrt{-a}=-a$.

Mr. Playfair makes $\sqrt{-1}\times\sqrt{1-z^2}=\sqrt{-1+z^2}$, which we conceive is not a correct result when z is more than 1.

Let $z=\sqrt{2}$, then $z^2=2$; whence, the above expression becomes $\sqrt{-1}\times\sqrt{1-2}=\sqrt{-1+2}$, or $\sqrt{-1}\times\sqrt{-1}=\sqrt{1}=1$, which result does not agree with his other position unless he takes $\sqrt{1}=-1$, which we know would be proper; that is, when z is more than 1, we have $\sqrt{-1}\times\sqrt{1-z^2}=-\sqrt{-1+z^2}$.

PROBLEM.

(241.) Multiply $\sqrt[4]{a}$ by $\sqrt{-1}$.

SOLUTION.

Since $\sqrt{-1}=(-1)^{\frac{1}{2}}=(-1)^{\frac{2}{4}}=\sqrt[4]{(-1)^2}$, we have, by the ordinary rule, $\sqrt[4]{(-1)^2}\times\sqrt[4]{a}=\sqrt[4]{1}\times\sqrt[4]{a}=\sqrt[4]{a}$. But $\sqrt[4]{a}=\pm\sqrt{\pm\sqrt{a}}$; that is, the 4th root of a is $+\sqrt{+\sqrt{a}}$, $+\sqrt{-\sqrt{a}}$, $-\sqrt{+\sqrt{a}}$, or $-\sqrt{-\sqrt{a}}$. Now, to which of these forms must $\sqrt[4]{a}$, in the present case, be made equal? We have

$$\sqrt[4]{(-1)^2}\times\sqrt[4]{a}=\sqrt{\sqrt{(-1)^2}}\times\sqrt{\sqrt{a}}=\sqrt{\sqrt{(-1)^2}\sqrt{a}}=\sqrt{-\sqrt{a}}.$$

Since $\sqrt{(-1)^2}=-1$. Therefore, $\sqrt{-1}\times\sqrt[4]{a}=\sqrt{-\sqrt{a}}$.

EXAMPLES.*

1. Multiply $2\sqrt{-2}$ by $3\sqrt{-3}$. *Ans.* $-6\sqrt{6}$.
2. Multiply $4\sqrt{-3}$ by $9\sqrt{-12}$. *Ans.* -216.
3. Multiply $-2\sqrt{-2}$ by $-3\sqrt{-3}$. *Ans.* $-6\sqrt{6}$.
4. Multiply $-2\sqrt{-2}$ by $3\sqrt{-3}$. *Ans.* $+6\sqrt{6}$.
5. Multiply $1+\sqrt{-1}$ by $1+\sqrt{-1}$. *Ans.* $2\sqrt{-1}$.
6. Multiply $1+\sqrt{-1}$ by $1-\sqrt{-1}$. *Ans.* 2.
7. Multiply $a+\sqrt{-b^2}$ by $a+\sqrt{-b^2}$. *Ans.* $a^2-b^2+2ab\sqrt{-1}$.
8. Multiply $5+2\sqrt{-3}$ by $2-\sqrt{-3}$. *Ans.* $16-\sqrt{-3}$.
9. Multiply $\sqrt[4]{-2}$ by $\sqrt[4]{-8}$. *Ans.* $2\sqrt{-1}$.
10. Multiply $2\sqrt[6]{-4}$ by $3\sqrt[6]{-16}$. *Ans.* -12.

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* A glance at these examples shows that the results are the same as would be obtained by the following

RULE.

To multiply one quadratic imaginary surd by another, multiply the quantities under the radical signs, according to the rule for signs given in multiplication, but the coefficients of the radicals, according to the reverse of that rule; that is, making the product of LIKE *signs* MINUS, *and unlike* PLUS.

This rule will not hold when but *one* of the quantities multiplied is an *imaginary* quadratic surd.

DIVISION OF IMAGINARY QUANTITIES.

PROBLEM

(242.) 1. Divide $\sqrt{-a}$ by $\sqrt{-b}$.

SOLUTION.

We have $\frac{\sqrt{-a}}{\sqrt{-b}}=\frac{\sqrt{a}\times\sqrt{-1}}{\sqrt{b}\times\sqrt{-1}}=\sqrt{\frac{a}{b}}$; since the imaginary quantities cancel.

PROBLEM

2. Divide $-\sqrt{-a}$ by $-\sqrt{-b}$.

SOLUTION.

$$\frac{-\sqrt{-a}}{-\sqrt{-b}}=\frac{-\sqrt{a}\times\sqrt{-1}}{-\sqrt{b}\times\sqrt{-1}}=+\sqrt{\frac{a}{b}}.$$

EXAMPLES.

1. Divide $6\sqrt{-3}$ by $2\sqrt{-4}$. *Ans.* $\frac{3}{2}\sqrt{3}$.

2. Divide $-\sqrt{-1}$ by $-6\sqrt{-3}$. *Ans.* $\frac{1}{18}\sqrt{3}$.

3. Divide $1+\sqrt{-1}$ by $1-\sqrt{-1}$. *Ans.* $\sqrt{-1}$.

4. Divide $4+\sqrt{-2}$ by $2-\sqrt{-2}$. *Ans.* $1+\sqrt{-2}$.

5. Divide 1 by $\sqrt{-1}$. *Ans.* $-\sqrt{-1}$.

6. Divide a by $\sqrt{a}\sqrt{-1}$. *Ans.* $-\sqrt{a}\sqrt{-1}$.

7. Divide $2\sqrt{8}-\sqrt{-10}$ by $-\sqrt{-2}$. *Ans.* $\sqrt{5}+4\sqrt{-1}$.

8. Divide $5-\sqrt{-2}$ by $1+\sqrt{-2}$. *Ans.* $1-2\sqrt{-2}$

9. Divide $14-\sqrt{15}-(7\sqrt{3}+2\sqrt{5})\sqrt{-1}$ by $7-\sqrt{-5}$.
Ans. $2-\sqrt{-3}$.

10. Divide a by $b\sqrt{-1}$. *Ans.* $-\frac{a}{b}\sqrt{-1}$.

INVOLUTION OF IMAGINARY QUANTITIES.

PROBLEM.

(243.) Cube $a-\sqrt{-b^2}$.

SOLUTION.

Calling $\sqrt{-b^2}=c$, we have $(a-c)^3=a^3-3a^2c+3ac^2-c^3$.

If now, we ascertain the value of c^2 and c^3, we can put these values for c^2 and c^3 in the above expression. The square of c, or of $\sqrt{-b^2}=b\sqrt{-1}$ is evidently $-b^2$, and $c^3=c^2c$, or $-b^2\sqrt{-b^2}=-b^3\sqrt{-1}$.

Hence, $(a-\sqrt{-b^2})^3=a^3-3a^2b\sqrt{-1}-3ab^2+b^3\sqrt{-1}=a^3-3ab^2+(b^3-3a^2b)\sqrt{-1}$.

PROBLEM.

(244.) To find formulas for the powers of $\sqrt{-1}$.

SOLUTION.

Let the index of the powers be $4n$, $4n+1$, $4n+2$, and $4n+3$, which comprise all positive integer numbers.

Let $a=\sqrt{-1}$, and we have

$$(\sqrt{-1})^{4n}=a^{4n}=(a^4)^n=(+1)^n=+1;$$
$$(\sqrt{-1})^{4n+1}=a^{4n+1}=a^{4n}.a=a=+\sqrt{-1}.$$
$$(\sqrt{-1})^{4n+2}=a^{4n+2}=a^{4n}.a^2=a^2=-1.$$
$$(\sqrt{-1})^{4n+3}=a^{4n+3}=a^{4n}\cdot a^3=-1.a=-\sqrt{-1}.$$

Thus, to obtain any power of $\sqrt{-1}$, it is only necessary to divide the exponent of the power by 4, and the power of $\sqrt{-1}$ indicated by the remainder will be the result required.

EXAMPLES.

1. Raise $\sqrt{-1}$ to the 66th power. *Ans.* -1.
2. Raise $\sqrt{-1}$ to the 103d power. *Ans.* $-\sqrt{-1}$.
3. Raise $\sqrt{-1}$ to the 400th power. *Ans.* $+1$.
4. Raise $a\sqrt{-1}$ to the $4n$th power. *Ans.* a^{4n}.
5. Raise $a\sqrt{-1}$ to the $4n+1$st power. *Ans.* $a^{4n+1}\sqrt{-1}$.
6. Raise $a\sqrt{-1}$ to the $4n+2$d power. *Ans.* $-a^{4n+2}$.

7. Raise $a\sqrt{-1}$ to the $4n+3$rd power. *Ans.* $-a^{4n+3}\sqrt{-1}$.

8. Square $a\pm\sqrt{-b}$. *Ans.* $a^2-b\pm 2a\sqrt{-b}$.

9. Cube $a\pm\sqrt{-b}$. *Ans.* $a^3-3ab\pm[(3a^2-b)\sqrt{b}]\sqrt{-1}$.

10. Raise $a+\sqrt{-1}$ to the 6th power.

Ans. $a^6-15a^4+15a^2-1+2(3a^4-10a^2+3)a\sqrt{-1}$.

EVOLUTION OF IMAGINARY QUANTITIES.

PROBLEM.

(245.) To extract the square root of a binomial surd, one of whose terms is rational and the other an imaginary quadratic surd.

SOLUTION.

Let $\sqrt{x}+\sqrt{-y}$ represent the square root of $a+\sqrt{-b}$. Then $x+2\sqrt{-xy}-y$, or $x-y+2\sqrt{-xy}=a+\sqrt{-b}$; whence, by Theorem. (**204**), we have $x-y=a$. We know by Theorem (**205**), that if $\sqrt{x}+\sqrt{-y}=\sqrt{a+\sqrt{-b}}$, that $\sqrt{x}-\sqrt{-y}=\sqrt{a-\sqrt{-b}}$. It is evident that the product of $\sqrt{x}+\sqrt{-y}$ by $\sqrt{x}-\sqrt{-y}$ which is $x+y$, must equal the product of $\sqrt{a+\sqrt{-b}}$ by $\sqrt{a-\sqrt{-b}}$, which is $\sqrt{a^2+b}$. This gives $x+y=\sqrt{a^2+b}$. If we extract the square root of half the sum of $x-y$ and $x+y$ the result must equal the square root of half the sum of a and $\sqrt{a^2+b}$; that is, $\sqrt{x}=\sqrt{\dfrac{a+\sqrt{a^2+b}}{2}}$. Also, if we extract the square root of half the difference of $x-y$ and $x+y$, the result must equal the square root of half the difference of a and $\sqrt{a^2+b}$; that is, $\sqrt{-y}=\sqrt{\dfrac{a-\sqrt{a^2+b}}{2}}$. Whence, we have the formula

$$\sqrt{a+\sqrt{-b}}=\sqrt{\frac{a+\sqrt{a^2+b}}{2}}+\sqrt{\frac{a-\sqrt{a^2+b}}{2}}.$$

Letting $\sqrt{a-\sqrt{-b}}=\sqrt{x}-\sqrt{-y}$, we would have

$$\sqrt{a-\sqrt{-b}}=\sqrt{\frac{a+\sqrt{a^2+b}}{2}}-\sqrt{\frac{a-\sqrt{a^2+b}}{2}}.$$

PROBLEM.

(246.) Extract the square root of $7+6\sqrt{-2}$.

SOLUTION.

Since, $7+6\sqrt{-2}=7+\sqrt{-72}$, we have by putting 7 for a, and 72 for b in the general formula,

$$\sqrt{7+\sqrt{-72}}=\sqrt{\frac{7+\sqrt{49+72}}{2}}+\sqrt{\frac{7-\sqrt{49+72}}{2}}=3+\sqrt{-2}.$$

We could also solve this by inspection. Thus, since $7+6\sqrt{-2}=9+2\cdot3\sqrt{-2}-2$, we see that twice the product of the square roots of 9 and -2 is equal to the second term, and therefore, $3+\sqrt{-2}$ is the square root.

EXAMPLES.

1. Extract the square root of $31+42\sqrt{-2}$. *Ans.* $7+3\sqrt{-2}$.

2. Extract the square root of $-3+\sqrt{-16}$. *Ans.* $1+2\sqrt{-1}$.

3. Extract the square root of $4\sqrt{-6}-2$. *Ans.* $2+\sqrt{-6}$.

4. Extract the square root of $2+4\sqrt{-42}$. *Ans.* $\sqrt{14}+2\sqrt{-3}$.

5. Extract the square root of $\sqrt{-1}$. *Ans.* $\frac{1}{2}\sqrt{2}+\frac{1}{2}\sqrt{-2}$.

6. Extract the square root of $-2-2\sqrt{-15}$. *Ans.* $\sqrt{3}-\sqrt{-5}$.

7. Extract the square root of $2cd\sqrt{-1}$. *Ans.* $(1+\sqrt{-1})\sqrt{cd}$.

8. Extract the square root of $8\sqrt{-1}$. *Ans.* $2+2\sqrt{-1}$.

9. Extract the square root of $-\sqrt{-1}$. *Ans.* $\frac{1}{2}\sqrt{2}-\frac{1}{2}\sqrt{-2}$.

10. Extract the square root of $\frac{a^2c}{b^2}-cd+\frac{ac\sqrt{4d}}{b}\sqrt{-1}$.

Ans. $\frac{a}{b}\sqrt{c}+\sqrt{-cd}$.

PROBLEM.

(247.) To extract the cube root of $a+\sqrt{-b}$.

BOMBELLI'S RULE.

First find $\sqrt[3]{a^2+b}$, *then, by trials, search out a number* c, *and a square root* $\sqrt{d}$, *such that the sum of their squares* c^2+d *be* $=\sqrt[3]{a^2+b}$, *and also,* c^3-3cd *be*$=a$; *then shall* $c+\sqrt{-d}$ *be the cube root of* $a+\sqrt{-b}$ *sought.*

PROBLEM.

(248.) Extract the cube root of $2+11\sqrt{-1}$.

SOLUTION.

Since, $2+11\sqrt{-1}=2+\sqrt{-121}$, we have $a=2$ and $b=121$; whence, $\sqrt[3]{a^2+b}=\sqrt[3]{125}=5$; then taking $c=2$, and $d=1$, we obtain $c^2+d=5=\sqrt[3]{a^2+b}$, and $c^3-3cd=8-6=2=a$, as it ought; therefore, $2+\sqrt{-1}$ is the cube root of $2+11\sqrt{-1}$. This may also be solved by Tartalea's rule. Thus 2 can be separated into two parts 8 and -6, one of which is a perfect cube, and the other divisible by 3. Therefore, $r^3=8$, or $r=2$ and $3s=-6$, or $s=-2$; whence, $r+\sqrt{\frac{s}{r}}=2+\sqrt{\frac{-2}{2}}$, or $2+\sqrt{-1}$, the same result as before.

EXAMPLES.

1. Extract the cube root of $2+2\sqrt{-1}$. *Ans.* $-1+\sqrt{-1}$.

2. Extract the cube root of $2-\sqrt{-121}$. *Ans.* $2-\sqrt{-1}$.

3. Extract the cube root of $81\pm30\sqrt{-3}$. *Ans.* $-3\pm2\sqrt{-3}$.

4. Extract the cube root of $-10+9\sqrt{-3}$. *Ans.* $2+\sqrt{-3}$.

5. Extract the cube root of $-5-\sqrt{-2}$. *Ans.* $1-\sqrt{-2}$.

6. Extract the cube root of $-4+10\sqrt{-2}$. *Ans.* $2+\sqrt{-2}$.

7. Extract the cube root of $9+25\sqrt{-2}$. *Ans.* $3+\sqrt{-2}$.

MODULUS.

(249.) The *modulus* of $a+\beta\sqrt{-1}$ is $+\sqrt{a^2+\beta^2}$, or the square root of the sum of the squares of a and β. Thus, $+\sqrt{16+9}=+5$ is the modulus of $4+3\sqrt{-1}$.

(250.) Two imaginary expressions are *conjugate*, when they differ only in the sign of $\sqrt{-1}$; as, $a+\beta\sqrt{-1}$, and $a-\beta\sqrt{-1}$.

These conjugate expressions have the same *modulus*.

THEOREM.

(251.) In order that $\alpha+\beta\sqrt{-1}$ be equal to zero, it is necessary and sufficient that its modulus $+\sqrt{\alpha^2+\beta^2}$ be equal to zero.

DEMONSTRATION.

If the modulus $\sqrt{\alpha^2+\beta^2}$ is not equal to zero, α and β will not be equal to zero at the same time, and, consequently, $\alpha+\beta\sqrt{-1}$ will not be equal to zero. But, if $\sqrt{\alpha^2+\beta^2}$ is equal to zero, α and β must each equal zero; whence, $\alpha+\beta\sqrt{-1}$ would also be equal to zero.

THEOREM.

(252.) The modulus of the product of two imaginary factors is equal to the product of their moduli.

DEMONSTRATION.

The product of $\alpha+\beta\sqrt{-1}$ and $\alpha'+\beta'\sqrt{-1}$ is $(\alpha\alpha'-\beta\beta')+(\alpha\beta'+\beta\alpha')\sqrt{-1}$, and the modulus of their product is, therefore, equal to $\sqrt{(\alpha\alpha'-\beta\beta')^2+(\alpha\beta'+\beta\alpha')^2}=\sqrt{\alpha^2\alpha'^2+\beta^2\beta'^2+\alpha^2\beta'^2+\beta^2\alpha'^2}=$ $\sqrt{\alpha^2(\alpha'^2+\beta'^2)+\beta^2(\alpha'^2+\beta'^2)}=\sqrt{(\alpha^2+\beta^2)(\alpha'^2+\beta'^2)}=\sqrt{\alpha^2+\beta^2}\times$ $\sqrt{\alpha'^2+\beta'^2}$. *Q. E. D.*

THEOREM.

(253.) The modulus of the quotient resulting from the division of one imaginary quantity by another is equal to the quotient of their muduli.

DEMONSTRATION.

Let $\alpha''+\beta''\sqrt{-1}$ represent the quotient of $\alpha+\beta\sqrt{-1}$ by $\alpha'+\beta'\sqrt{-1}$, since it can be easily proved that the quotient must be of this form. Hence, we have,

$$\alpha+\beta\sqrt{-1}=(\alpha'+\beta'\sqrt{-1})(\alpha''+\beta''\sqrt{-1}).$$

By the last Theorem, we have $\sqrt{\alpha^2+\beta^2}=\sqrt{\alpha'^2+\beta'^2}.\sqrt{\alpha''^2+\beta''^2}$.

We may consider $\sqrt{\alpha''^2+\beta''^2}$ as the quotient arising from dividing $\sqrt{\alpha^2+\beta^2}$ by $\sqrt{\alpha'^2+\beta'^2}$; that is, $\sqrt{\alpha''^2+\beta''^2}=\dfrac{\sqrt{\alpha^2+\beta^2}}{\sqrt{\alpha'^2+\beta'^2}}$. *Q. E. D.*

THEOREM.

(254.) In order that the product of imaginary factors be equal to zero, it is necessary and sufficient that one of the factors be equal to zero.

DEMONSTRATION.

It can be easily shown that the product of two or more imaginary factors must be of the form $\alpha+\beta\sqrt{-1}$.

In order that this product may be equal to zero, we have seen that its modulus must be equal to zero; but this modulus is the product of the moduli of the several factors, and these moduli are real quantities, and consequently their product can not be zero, unless at least one of these moduli be also equal to zero; but when a modulus is equal to zero, its corresponding imaginary factor must also be equal to zero. Hence, the Theorem is true.

MISCELLANEOUS EXAMPLES IN RADICALS.

1. Extract the cube root of $20+14\sqrt{2}$. *Ans.* $2+\sqrt{2}$.

2. Simplify $\dfrac{13\sqrt{10}-40}{22-7\sqrt{10}}$. *Ans.* $-(5+\sqrt{10})$.

3. Extract the square root of $-1+4\sqrt{-5}$. *Ans.* $2+\sqrt{-5}$.

4. Simplify $3\sqrt[6]{4a^2}+2\sqrt[3]{2a}$. *Ans.* $5\sqrt[3]{2a}$.

5. Simplify $\sqrt[3]{8a^3b+16a^4}-\sqrt[3]{b^4+2ab^3}$. *Ans.* $(2a-b)\sqrt[3]{2a+b}$.

6. Simplify $x\sqrt[3]{\dfrac{8a^4}{27b^3}+\dfrac{16a^3}{27b^2}}$. *Ans.* $\dfrac{2ax}{3b}\sqrt[3]{a+2b}$.

7. Simplify $\sqrt{4a^5b^2-20a^3b^3+25ab^4}$. *Ans.* $(2a^2-b)b\sqrt{a}$.

8. Simplify $\sqrt[3]{81}-2\sqrt[3]{24}+\sqrt{28}+2\sqrt{63}$. *Ans.* $8\sqrt{7}-\sqrt[3]{3}$.

9. Simplify $\sqrt{12}+2\sqrt{27}+3\sqrt{75}+9\sqrt{48}$. *Ans.* $59\sqrt{3}$.

10. Simplify $\dfrac{\sqrt{a^2x-2ax^2+x^3}}{\sqrt{a^2+2ax+x^2}}$. *Ans.* $\dfrac{a-x}{a+x}\sqrt{x}$.

11. Simplify $\dfrac{a-b}{a+b}\cdot\dfrac{\sqrt{ac}}{\sqrt{a^2-2ab+b^2}}$. *Ans.* $\dfrac{\sqrt{ac}}{a+b}$.

12. Simplify $\frac{a+b}{a-b}\sqrt{\frac{a-b}{a+b}}$. *Ans.* $\sqrt{\frac{a+b}{a-b}}$.

13. Simplify $\sqrt[3]{2} \times \sqrt[6]{\tfrac{1}{3}} \times \sqrt[8]{3}$. *Ans.* $\sqrt[24]{\tfrac{256}{3}}$.

14. Simplify $\sqrt[5]{4} \times \sqrt[10]{3} \times \sqrt[15]{6}$. *Ans.* $\sqrt[30]{3981312}$.

15. Find the sum of $\sqrt[3]{\tfrac{1}{4}}$ and $\sqrt[3]{\tfrac{1}{32}}$. *Ans.* $\tfrac{3}{4}\sqrt[3]{2}$.

16. Find the difference of $\sqrt[3]{\tfrac{3}{5}}$ and $\sqrt[3]{\tfrac{25}{9}}$. *Ans.* $\tfrac{2}{15}\sqrt[3]{75}$.

17. Find a factor that will make a^{-3} rational. *Ans.* a^4.

18. Simplify $\frac{10}{\sqrt{5}+\sqrt{3}}$. *Ans.* $5\sqrt{5}-5\sqrt{3}$.

19. Extract the square root of $bc+2b\sqrt{bc-b^2}$.

Ans. $\pm(b+\sqrt{bc-b^2})$.

20. Extract the square root of $np+2m^2-2m\sqrt{np+m^2}$.

Ans. $\pm(\sqrt{np+m^2}-m)$.

21. Simplify $\sqrt{16+30\sqrt{-1}}+\sqrt{16-30\sqrt{-1}}$.

Ans. $\pm 6\sqrt{-1}$.

22. Reduce $\sqrt{16+30\sqrt{-1}}+\sqrt{16-30\sqrt{-1}}$ to its simplest form.

Ans. ± 10.

23. Simplify $\sqrt{31+12\sqrt{-5}}+\sqrt{-1+4\sqrt{-5}}$.

Ans. $\pm 8 \pm 2\sqrt{-5}$.

24. Reduce $\sqrt{31+12\sqrt{-5}}+\sqrt{-1+4\sqrt{-5}}$ to its simplest form.

Ans. ± 4.

25. Simplify $\sqrt{bc+2b\sqrt{bc-b^2}}+\sqrt{bc-2b\sqrt{bc-b^2}}$.

Ans. $\pm 2\sqrt{bc-b^2}$.

26. Reduce $\sqrt{bc+2b\sqrt{bc-b^2}}+\sqrt{bc-2b\sqrt{bc-b^2}}$ to its simplest form. *Ans.* $\pm 2b$.

27. Simplify $\left(\sqrt{\frac{1+x}{1-x}}-\sqrt{\frac{1-x}{1+x}}\right)\div\left(\sqrt{\frac{1+x}{1-x}}+\sqrt{\frac{1-x}{1+x}}\right)$. *Ans.* x.

28. Divide $2\sqrt{3} \times \sqrt[3]{4}$ by $\tfrac{1}{2}\sqrt[4]{2} \times \sqrt[3]{3}$. *Ans.* $4\sqrt[12]{288}$.

29. Divide $\frac{1}{2}\sqrt{\frac{1}{2}}$ by $\sqrt{2}+3\sqrt{\frac{1}{2}}$. *Ans.* $\frac{1}{10}$.

30. Multiply $\sqrt{2}\times\sqrt[3]{3}$ by $\sqrt[4]{\frac{1}{2}}\times\sqrt[3]{\frac{1}{3}}$. *Ans.* $\sqrt[12]{8}$.

31. Multiply $\sqrt[7]{\frac{4}{3}}\times\sqrt[3]{\frac{1}{2}}$ by $\sqrt[14]{6}$. *Ans.* $\sqrt[42]{\frac{2}{27}}$.

32. Divide $\sqrt{\sqrt{\frac{1}{2}}\times 2\sqrt[3]{3}}$ by $\sqrt{4\sqrt[3]{2}\times\sqrt{3}}$. *Ans.* $\frac{1}{2}\sqrt[12]{\frac{2}{3}}$.

33. Divide 1 by $\sqrt[4]{a}+\sqrt[4]{b}$.

Ans. $\dfrac{\sqrt[4]{a^3}-\sqrt[4]{a^2b}+\sqrt[4]{ab^2}-\sqrt[4]{b^3}}{a-b}$.

34. Multiply $4\sqrt{\frac{7}{3}}+5\sqrt{\frac{1}{2}}$ by $\sqrt{\frac{7}{3}}+2\sqrt{\frac{1}{2}}$. *Ans.* $\frac{43}{3}+\frac{13}{6}\sqrt{42}$.

35. Divide $\sqrt[4]{a}+\sqrt[4]{b}$ by $\sqrt[4]{a}-\sqrt[4]{b}$.

Ans. $\dfrac{a+b+2\sqrt{ab}+2\sqrt[4]{a^3b}+2\sqrt[4]{ab^3}}{a-b}$.

36. Cube $\dfrac{-1\pm\sqrt{-3}}{2}$. *Ans.* 1.

37. Simplify $\dfrac{2(2)^{\frac{1}{2}}\times\sqrt[3]{3}}{\frac{1}{2}\sqrt{2}}$. *Ans.* $4\cdot\sqrt[3]{3}$.

38. Simplify $\left\{\dfrac{\frac{1}{2}(2)^{\frac{1}{2}}\sqrt[3]{3}}{2\sqrt[4]{2}(3)^{\frac{1}{3}}}\right\}^4$. *Ans.* $\dfrac{1}{384}\cdot\sqrt[3]{3}$.

39. Simplify $\sqrt{\left\{\dfrac{(\frac{1}{2})^3+\sqrt{3\frac{1}{2}}}{2\sqrt{2}\cdot(\frac{3}{4})^{\frac{1}{2}}}\right\}^{\frac{1}{2}}}$. *Ans.* $\sqrt[4]{\frac{1}{6}(\frac{1}{8}\sqrt{6}+\sqrt{21})}$.

40. Find the sum of $5\sqrt{2}-1$ and $\dfrac{3\sqrt{2}+2}{5-3\sqrt{2}}$. *Ans.* $3+8\sqrt{2}$.

41. Find the difference between $\dfrac{\sqrt{14}+\sqrt{12}}{\sqrt{7}-\sqrt{6}}$ and $\dfrac{\sqrt{6}-\sqrt{4}}{\sqrt{3}+\sqrt{2}}$.

Ans. $8\sqrt{2}+4\sqrt{21}+4\sqrt{3}$.

42. Divide $2\sqrt{8}\times 4\sqrt[4]{\frac{1}{4}}$ by $4\sqrt[3]{\frac{1}{2}}\times 4\sqrt[6]{4}$. *Ans.* 1.

43. Extract the square root of $7-\sqrt{13}$. *Ans.* $\frac{1}{2}\sqrt{26}-\frac{1}{2}\sqrt{2}$.

44. Extract the square root of $\frac{3}{4}+\frac{5}{9}\sqrt[3]{\frac{1}{5}}-\frac{1}{3}\sqrt[6]{675}$.

Ans. $\frac{1}{2}\sqrt{3}-\frac{1}{3}\sqrt[3]{5}$.

CHAPTER X.

EQUATIONS.

(255.) An *equation* is an algebraic statement denoting the equality, in value, of two algebraic expressions; as, $ax+b=c$, $6x^2+2x=7$, and $ax-b=0$.

(256.) The *absolute term* of an equation, is that term which is completely known, or is considered as known, and which is equal to the sum of all the unknown terms. Thus, in the equations $6x=9$, $4x^2+3x-7=0$, and $ax=b-c$; 9, 7, and $(b-c)$ are the *absolute terms.*

(257.) The *first, or left hand member* of an equation, is the part of the equation which precedes the sign of equality.

(258.) The *second or right hand member* of an equation is the part of the equation which follows the sign of equality.

THEOREM.

(259.) *Any term of an equation may be transposed from one member to the other by changing its sign.*

DEMONSTRATION.

Let $ax+b=c$ be an equation. This equation may assume the following form $ax+b=c+b-b$. It is evident that we should still have a correct equation if we should omit $+b$ in both members. Thus $ax=c-b$.

Again, let $ax-b=d$. Because $ax-b=d+b-b$, we have $ax=d+b$. These results show that if we transpose any term of an equation from one member to the other, its sign must be changed in order not to destroy the equality.

The truth of this Theorem may be proved as follows:

Letting $ax+b=c$ and subtracting $+b$ from both members, we have

$$\begin{array}{r} ax+b=c \\ +b=b \\ \hline ax \quad =c-b, \end{array}$$

or by adding $-b$ to both members, we have

$$\begin{array}{l} ax+b=c \\ \quad\;\, -b=-b \\ \hline ax \qquad =c-b. \end{array}$$

Letting $ax-b=d$ and subtracting $-b$ from both members, we have

$$\begin{array}{l} ax-b=d \\ \quad\;\, -b=-b \\ \hline ax \qquad =d+b, \end{array}$$

or by adding $+b$ to both members, we have

$$\begin{array}{l} ax-b=d \\ \quad\;\, +b=+b \\ \hline ax \qquad =d+b \end{array}$$

PROBLEM.

(260.) Transpose all the known terms of $x^3+6x^2-4x-4+d-b=0$ to the second member.

SOLUTION.

Transposing -4, $+d$, $-b$ by changing their signs, or by adding $+4-d+b$ to both members, or by subtracting $-4+d-b$ from both members, we have $x^3+6x^2-4x=4-d+b$.

EXAMPLES.

1. Transpose in $6x+4=4x+8$ the terms $+4$, and $4x$.

Ans. $2x=4$.

2. Transpose in $5x^2-2x+3=4x^2-4x+1$ the terms $+3$, $4x^2$, and $-4x$.

Ans. $x^2+2x=-2$.

3. Transpose in $\frac{2}{x}-\frac{1}{y}=\frac{1}{x}+\frac{1}{y}$ the terms $-\frac{1}{y}$ and $\frac{1}{x}$. *Ans.* $\frac{1}{x}=\frac{2}{y}$.

4. Transpose in $\frac{4}{x+a}+\frac{3}{x+y}=\frac{3}{x+a}+\frac{4}{x+y}$ the terms $\frac{3}{x+y}$ and $\frac{3}{x+a}$.

Ans. $\frac{1}{x+a}=\frac{1}{x+y}$.

5. Transpose in $\frac{4x+4}{5x+b}-\frac{a-b}{x-d}+6=\frac{2x+1}{5x+b}+\frac{a+b}{x-d}+5$, the term $-\frac{a-b}{x-d}$, $+6$ and $\frac{2x+1}{5x+b}$.

Ans. $\frac{2x+3}{5x+b}=\frac{2a}{x-d}-1$.

6. Transpose in $7x^4-6x^2+6=-x^2+8$ the terms $+6$, and $-x^2$.

Ans. $7x^4-5x^2=2$.

7. Transpose in $4\sqrt{x}+\sqrt{y}=\sqrt{x}+3\sqrt{y}$ the terms $\sqrt{y}$ and $\sqrt{x}$.

Ans. $3\sqrt{x}=2\sqrt{y}$.

8. Transpose in $\frac{2\sqrt{3}+3\sqrt{2}}{2x}+4=\frac{2\sqrt{3}-3\sqrt{2}}{2x}+9$ the terms $+4$, and $\frac{2\sqrt{3}-3\sqrt{2}}{2x}$.

Ans. $\frac{3\sqrt{2}}{x}=5$.

9. Transpose in $a\sqrt{x}+c\sqrt{d}=b\sqrt{x}+b\sqrt{d}$ the term $c\sqrt{d}$ and $b\sqrt{x}$.

Ans. $(a-b)\sqrt{x}=(b-c)\sqrt{d}$.

10. Transpose in $\frac{9x+20}{36}=\frac{4x-12}{5x-4}+\frac{x}{4}$ the term $\frac{x}{4}$.

Ans. $\frac{5}{9}=\frac{4x-12}{5x-4}$.

11. Transpose in $\frac{7x+16}{21}-\frac{x+8}{4x-11}=\frac{x}{3}$ the terms $\frac{x+8}{4x-11}$ and $\frac{x}{3}$.

Ans. $\frac{16}{21}=\frac{x+8}{4x-11}$.

12. Transpose in $\frac{20x}{25}+\frac{36}{25}+\frac{5x+20}{9x-16}=\frac{4x}{5}+\frac{86}{25}$ the terms $\frac{36}{25}$ and $\frac{4x}{5}$.

Ans. $\frac{5x+20}{9x-16}=2$.

THEOREM.

(261.) *Any equation containing fractions may be cleared of these fractions by multiplying both members by the least common multiple of the denominators of the fractions.*

DEMONSTRATION.

Let $\frac{ax^2}{b}+\frac{cx}{ab^2}=\frac{d}{a^2}+\frac{c}{a}+b$ be an equation containing fractions.

The least common multiple of the denominators is a^2b^2:

Multiplying both members of the equation then by a^2b^2, we have $a^3bx^2+acx=b^2d+ab^2c+a^2b^3$, an equation which contains no fractions.

It is evident that the same method of proceeding would produce an equation containing no fractions, whatever might be the number of fractions in the equation to be cleared.

Corollary.—To clear an equation of fractions, we may multiply by *any* common multiple of the denominators.

PROBLEM

(262.) 1. Clear $\frac{1}{4}x^2+\frac{1}{6}x=8$ of fractions.

SOLUTION.

The least common multiple of the denominators is 12.

Multiplying both members by 12, we have $3x^2+2x=96$.

We might also, multiply by 24, or any multiple of 12.

PROBLEM

2. Clear $x+\frac{x}{2}+\frac{3x}{4}+\frac{2x}{7}+\frac{x}{14}=146$ of fractions.

SOLUTION.

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Multiplying both members by 56, the least common multiple of the denominators, we have $56x+28x+42x+16x+4x=146\cdot 56$.

PROBLEM

3. Clear $\frac{3x}{4}-\frac{x-1}{2}=6x-\frac{20x+13}{4}$ of fractions.

SOLUTION.

Multiplying by 4, we obtain $3x-2x+2=24x-20x-13$.

It must be carefully noted that in the numerators $x-1$, and $20x+13$, the x and $20x$ are both positive, the negative sign belonging to the whole fraction.

Corollary.—Hence, we see that when a denominator is removed from a negative fraction by multiplication or division, every term of the resulting numerator must have its sign changed.

PROBLEM

4. Clear $x-\frac{2y-x}{23-x}=20-\frac{59-2x}{2}$ of fractions.

SOLUTION.

The least common multiple of the denominators is $2(23-x)=46-2x$. But, in order to save labor, we shall first multiply by 2, and then by $23-x$.

Multiplying by 2, we get $2x-\frac{4y-2x}{23-x}=40-59+2x$.

Let us simplify this equation before multiplying by $23-x$. This can be done by dropping $2x$ from both members, and adding 40 and -59 together, whence results $-\frac{4y-2x}{23-x}=-19$.

Now multiplying both members by $23-x$, we obtain $-4y+2x=-19\cdot 23+19x$, or $2x-4y=19x-437$.

EXAMPLES.

1. Clear $\frac{x}{5}=\frac{y}{5}+2$ of fractions. *Ans.* $x=y+10$.

2. Clear $5x-\frac{y}{3}=\frac{3x}{4}-\frac{8y}{6}+37$ of fractions.

Ans. $60x-4y=9x-16y+444$.

3. Clear $4-\frac{y-x}{6}=y-17\frac{2}{3}$ of fractions.

Ans. $24-y+x=6y-106$.

4. Clear $\frac{7x-21}{6}+y-\frac{x}{3}=4+\frac{3x-19}{2}$ of fractions.

Ans. $7x-21+6y-2x=24+9x-57$.

5. Clear $\frac{2x+y}{2}-\frac{9x-7}{8}=\frac{3y+9}{4}-\frac{4x+5y}{16}$ of fractions.

Ans. $16x+8y-18x+14=12y+36-4x-5y$.

6. Clear $\frac{1}{3}(x+y)+6=y$ of fractions. *Ans.* $x+y+18=3y$.

7. Clear $5x-\frac{3x-3}{x-3}=2x+\frac{3x-6}{2}$ of fractions.

Ans. $10x^2-30x-6x+6=4x^2-12x+3x^2-15x+18$.

8. Clear $\frac{x}{x+1}+\frac{x+1}{x}=\frac{13}{6}$ of fractions.

Ans. $6x^2+6x^2+12x+6=13x^2+13x$.

9. Clear $\frac{2\sqrt{x}+2}{4+\sqrt{x}}=\frac{4-\sqrt{x}}{\sqrt{x}}$ of fractions. *Ans.* $2x+2\sqrt{x}=16-x$.

10. Clear $\sqrt{\frac{x+a}{x}}+2\sqrt{\frac{a}{x+a}}=b^2\sqrt{\frac{x}{x+a}}$ of fractions.

Ans. $x+a+2\sqrt{ax}=b^2x$.

SIMPLE EQUATIONS.

(263.) A SIMPLE EQUATION is one in which the exponents of the unknown terms are ± 1, or $\pm$, a proper fraction whose numerator is one; as, $6x+a=b$, $7x^{\frac{1}{2}}+y^{\frac{1}{3}}=8$, $x^{-1}+c=d$, $4x^{-\frac{1}{6}}+5=9$, &c.

SIMPLE EQUATIONS CONTAINING ONE UNKNOWN QUANTITY.

PROBLEM.

(264.) To solve a simple equation containing but one unknown quantity, that is, to reduce it to its simplest form.

RULE.

1. *Collect the terms as much as possible by addition and transposition.*

2. *Clear the equation of fractions.*

3. *Transpose all the unknown terms to the first member, and the known to the second member.*

4. *Collect all the unknown term, when not already done, into one term, and put the second member in its simplest form.*

5. *Divide both members of the equation by the coefficient of the unknown quantity, and the resulting equation will be the simplest form of the given equation, and will indicate the value of the unknown quantity.*

DEMONSTRATION.

It is evident that the operations indicated in the rule are such as will not destroy the truth of the equation, nor will they change the value of the unknown quantity. Hence, the value of the unknown quantity in the final equation will always be the same as in the equation from which it results.

PROBLEM

(265.) 1. Solve $3x-4=7x-16$. (1).

SOLUTION.

$-4x=-12$ (2) $=Eq.(1)$ transposed and added.

$x=3.$ (3) $=(2)\div -4$.

PROBLEM

2. Solve $ax-c=d-bx$. (1).

SOLUTION.

$ax+bx=c+d$ (2) $=(1)$ transposed.

$(a+b)x=c+d$ (3) $=(2)$ added.

$x=\dfrac{c+d}{a+b}$ (4) $=(3)\div(a+b)$.

PROBLEM

3. Solve $ax^2+bx=9x^2+cx$. (1).

SOLUTION.

$$\begin{aligned} ax+\ b&=9x+c && (2)=(1)\div x.\\ ax-9x&=\ c-b && (3)=(2)\text{ transposed.}\\ (a-9)x&=\ c-b && (4)=(3)\text{ added.}\\ x&=\frac{c-b}{a-9} && (5)=(4)\div(a-9). \end{aligned}$$

PROBLEM

4. Solve $\frac{20x}{25}+\frac{36}{25}+\frac{5x+20}{9x-16}=\frac{4x}{5}+\frac{86}{25}$.

SOLUTION.

This equation may be put in the following form,
$\frac{4x}{5}+\frac{36}{25}+\frac{5x+20}{9x-16}=\frac{4x}{5}+2+\frac{36}{25}$. (1). Dropping equals from both members, we get $\frac{5x+20}{9x-16}=2$ (2).

$$\begin{aligned} 5x+20&=18x-32 && (3)=(2)\text{ cleared of fractions.}\\ -13x&=-52 && (4)=(3)\text{ transposed.}\\ x&=3 && (5)=(4)\div-13. \end{aligned}$$

PROBLEM

5. Solve $11x+\frac{2x}{3}+\frac{5x}{4}+\frac{x}{6}+\frac{x}{24}=315$. (1).

SOLUTION.

$$\begin{aligned} 264x+16x+30x+4x+x&=315\times24 && (2)=(1)\times24.\\ 315x&=315\times24 && (3)=(2)\text{ added.}\\ x&=24\,. && (4)=(3)\div315. \end{aligned}$$

REMARK.—No multiplication should be performed unless there is a necessity for it. If we had multiplied 315 by 24, we would in this case have performed work which was unnecessary.

PROBLEM

6. Solve $\sqrt{4x+16}=12$. (1).

SOLUTION.

We must first free this equation of the radical, which can be done by squaring both members. We learned in Involution that to square the square root of a quantity, we have only to omit the radical.

$$4x+16=144 \qquad (2)=(1)^2.$$
$$4x=128 \qquad (3)=(2) \text{ transposed.}$$
$$x=32 \qquad (4)=(3)\div 4.$$

PROBLEM

7. Solve $\sqrt{12+x}=2+\sqrt{x}$. (1).

SOLUTION.

$$12+x=4+4\sqrt{x}+x \qquad (2)=(1)^2.$$
$$8=4\sqrt{x} \qquad (3)=(2) \text{ transposed.}$$
$$2=\sqrt{x} \qquad (4)=(3)\div 4.$$
$$4=x \qquad (5)=(4)^2.$$

PROBLEM

8. Given $\sqrt{x-a}=\sqrt{x}-\frac{1}{2}\sqrt{a}$ (1) to find the value of x.

SOLUTION.

$$x-a=x-\sqrt{ax}+\frac{a}{4} \qquad (2)=(1)^2.$$
$$\sqrt{ax}=\frac{5a}{4}$$
$$ax=\frac{25a^2}{16}$$
$$x=\frac{25a}{16}.$$

PROBLEM

9. Solve $\frac{\sqrt{x}+28}{\sqrt{x}+4}=\frac{\sqrt{x}+38}{\sqrt{x}+6}$. (1).

SOLUTION.

$$x+34\sqrt{x}+168=x+42\sqrt{x}+152 \qquad (2)=(1)\times(\sqrt{x}+4)(\sqrt{x}+6).$$
$$16=8\sqrt{x} \qquad (3)=(2) \text{ transposed.}$$
$$2=\sqrt{x} \qquad (4)=(3)\div 8.$$
$$4=x \qquad (5)=(4)^2.$$

PROBLEM

10. Solve $\sqrt[2]{2x+3}+4=7$.(1).

SOLUTION.

$$\sqrt[3]{2x+3}=3 \qquad (2)=(1) \text{ transposed.}$$
$$2x+3=27 \qquad (3)=(2)^3.$$
$$2x=24.$$
$$x=12.$$

PROBLEM

11. Given $\frac{1}{x}+\frac{1}{a}=\sqrt{\frac{1}{a^2}+\sqrt{\frac{4}{a^2x^2}+\frac{9}{x^4}}}$ (1) to find the value of x.

SOLUTION.

$$\frac{1}{x^2}+\frac{2}{ax}+\frac{1}{a^2}=\frac{1}{a^2}+\sqrt{\frac{4}{a^2x^2}+\frac{9}{x^4}} \qquad (2)=(1)^2.$$
$$\frac{1}{x^2}+\frac{2}{ax}=\sqrt{\frac{4}{a^2x^2}+\frac{9}{x^4}} \qquad (3)=(2) \text{ transposed.}$$
$$\frac{1}{x}+\frac{2}{a}=\sqrt{\frac{4}{a^2}+\frac{9}{x^2}} \qquad (4)=(3)\times x.$$
$$\frac{1}{x^2}+\frac{4}{ax}+\frac{4}{a^2}=\frac{4}{a^2}+\frac{9}{x^2} \qquad (5)=(4)^2.$$
$$\frac{4}{ax}=\frac{8}{x^2} \qquad (6)=(5) \text{ transposed.}$$
$$\frac{1}{a}=\frac{2}{x} \qquad (7)=(6)\times\frac{x}{4}.$$
$$x=2a \qquad (8)=(7)\times ax.$$

PROBLEM

12. Given $\frac{\sqrt{a}-\sqrt{a-x}}{\sqrt{a}+\sqrt{a-x}}=a$ (1) to find the value of x.

SOLUTION.

Rendering the denominator of the fraction which forms the first member rational by multiplying both numerator and denominator by $\sqrt{a}-\sqrt{a-x}$, the equation becomes

$$\frac{2a-2\sqrt{a^2-ax}-x}{x}=a \qquad (2).$$
$$-(a+1)x+2a=\sqrt{4a^2-4ax} \qquad (3)=(2)\times x \text{ and trans.}$$
$$(a+1)^2x^2-(4a^2+4a)x+4a^2=4a^2-4ax \qquad (4)=(3)^2$$
$$(a+1)^2x-4a^2-4a=-4a \qquad (5)=(4)\div x$$
$$(a+1)^2x=4a^2$$
$$x=\frac{4a^2}{(a+1)^2}=\left(\frac{2a}{a+1}\right)^2.$$

PROBLEM

13. Solve $\sqrt{x+2}+\sqrt{x}=\frac{4}{\sqrt{x+2}}$. (1)

SOLUTION.

$$x+2+\sqrt{x^2+2x}=4 \qquad (2)=(1)\times\sqrt{x+2}$$
$$\sqrt{x^2+2x}=2-x$$
$$x^2+2x=4-4x+x^2$$
$$6x=4$$
$$x=\tfrac{4}{6}=\tfrac{2}{3}.$$

PROBLEM

14. Solve $5-\frac{3x+25}{4}-\frac{17-6x}{9}=2\frac{1}{36}+x-\frac{9x+40}{8}$. (1)

SOLUTION.

$$20-3x-25-\frac{68-24x}{9}=8\tfrac{1}{9}+4x-\frac{9x+40}{2} \qquad (2)=(1)\times 4$$
$$-\frac{68-24x}{9}=13\tfrac{1}{9}+7x-\frac{9x+40}{2} \qquad (3)=(2) \text{ transposed.}$$
$$-68+24x=118+63x-\frac{81x+360}{2} \qquad (4)=(3)\times 9$$
$$\frac{81x+360}{2}=39x+186 \qquad (5)=(4) \text{ transposed.}$$
$$81x+360=78x+372$$
$$3x=12$$
$$x=4.$$

PROBLEM

15. Given $\frac{5x+4}{2}:\frac{18-x}{4}::7:4$ to find the value of x.

SOLUTION.

In a proportion, we have the first term divided by the second, equal to the third term divided by the fourth.

By performing these divisions, we have

$$\frac{10x+8}{18-x}=\frac{7}{4}$$
$$40x+32=126-7x$$
$$47x=94$$
$$x=2$$

PROBLEM

16. Given $\frac{4x+3}{6x-43}:1::2x+19:3x-19$ to find the value of x.

SOLUTION.

Putting this proportion in the form of an equation, we have

$$\frac{4x+3}{6x-43}=\frac{2x+19}{3x-19}.$$ Clearing of fractions,

we have $$12x^2+9x-76x-57=12x^2+114x-86x-817$$
$$-95x=-760$$
$$x=8$$

PROBLEM

17. Given $(a+x)(b+x)-a(b+c)=\frac{a^2c}{b}+x^2 (1)$ to find the value of x.

SOLUTION.

$$ab+(a+b)x+x^2-ab-ac=\frac{a^2c}{b}+x^2 \qquad (2)=(1) \text{ expanded.}$$

$$(a+b)x=\frac{a^2c}{b}+ac \qquad (3)=(2) \text{ transposed.}$$

$$(a+b)x=\frac{a^2c+abc}{b} \qquad (4)=(3) \left\} \begin{array}{l}\text{with terms of 2d}\\ \text{member added.}\end{array}\right.$$

$$(a+b)x=\frac{ac}{b}(a+b) \qquad (5)=(4) \text{ factored.}$$

$$x=\frac{ac}{b} \qquad (6)=(5)\div(a+b).$$

EXAMPLES.

1. Given $8x+7=52-7x$ to find the value of x. *Ans.* $x=3$.
2. Given $18x-13=6x+35$ to find the value of x. *Ans.* $x=4$.
3. Given $19x+13=59-4x$ to find the value of x. *Ans.* $x=2$.
4. Given $3x+4-\frac{x}{3}=46-2x$ to find the value of x.

   Ans. $x=9$.
5. Given $x^2+15x=35x-3x^2$ to find the value of x. *Ans.* $x=5$.
6. Given $4x+36=5x+34$ to find the value of x. *Ans.* $x=2$.
7. Given $3x^2-10x=8x+x^2$ to find the value of x. *Ans.* $x=9$.
8. Given $3ax-4ab=2ax-6ac$ to find the value of x.

   Ans. $x=4b-6c$.

9. Given $ax^2+abx=cdx$ to find the value of x.

Ans. $x=\dfrac{cd-ab}{a}$.

10. Given $x-7=\dfrac{x}{5}+\dfrac{x}{3}$ to find the value of x. Ans. $x=15$.

11. Given $\dfrac{x}{2}+\dfrac{x}{3}=\dfrac{x}{4}+7$ to find the value of x. Ans. $x=12$.

12. Given $\dfrac{x-5}{4}+6x=\dfrac{284-x}{5}$ to find the value of x.

Ans. $x=9$

13. Given $x+\dfrac{11-x}{3}=\dfrac{19-x}{2}$ to find the value of x. Ans. $x=5$.

14. Given $3x+\dfrac{2x+6}{5}=5+\dfrac{11x-37}{2}$ to find the value of x.

Ans. $x=7$.

15. Given $21+\dfrac{3x-11}{16}=\dfrac{5x-5}{8}+\dfrac{97-7x}{2}$ to find the value of x.

Ans. $x=9$.

16. Given $\dfrac{6x-4}{3}-2=\dfrac{18-4x}{3}+x$ to find the value of x.

Ans. $x=4$.

17. Given $\dfrac{x}{6}-\dfrac{x}{4}+10=\dfrac{x}{3}-\dfrac{x}{2}+11$ to find the value of x.

Ans. $x=12$.

18. Given $\dfrac{x+1}{5}+3=\dfrac{2x-3}{3}$ to find the value of x. Ans. $x=9$.

19. Given $\dfrac{7x+2}{3}+5x=28+\dfrac{5x-6}{7}$ to find the value of x.

Ans. $x=4$.

20. Given $\dfrac{3x+4}{5}+2x=\dfrac{22-x}{5}+16$ to find the value of x.

Ans. $x=7$.

21. Given $\dfrac{7x-8}{11}+\dfrac{15x+8}{13}=3x-\dfrac{31-x}{2}$ to find the value of x.

Ans. $x=9$.

22. Given $4x-\dfrac{19+2x}{5}=15-\dfrac{7x+11}{4}$ to find the value of x.

Ans. $x=3$.

23. Given $x+\frac{27-9x}{4}-\frac{5x+2}{6}=\frac{61}{12}-\frac{2x+5}{3}-\frac{29+4x}{12}$ to find the value of x.
Ans. $x=5$.

24. Given $\frac{31+4x}{3}-\frac{3x+47}{8}-\frac{3x-19}{16}=47\frac{3}{4}+\frac{16-10x}{11}-\frac{5x+20}{7}$ to find the value of x.
Ans. $x=17$.

25. Given $\frac{3a+x}{x}-5=\frac{6}{x}$ to find the value of x. *Ans.* $x=\frac{3a-6}{4}$.

26. Given $\frac{\frac{155}{8}x-13\frac{1}{2}}{11}-\frac{2}{3}\left(\frac{x+13\frac{7}{20}}{7}\right)=\frac{15}{14}(x-1)$ to find the value of x.
Ans. $x=2\frac{2}{5}$.

27. Given $\frac{4x-17}{9}-\frac{3\frac{2}{3}-22x}{33}=x-\frac{6}{x}\left\{1-\frac{x^2}{54}\right\}$ to find the value of x.
Ans. $x=3$.

28. Given $\frac{1}{2}\left\{\frac{2}{3}x+4\right\}-\frac{7\frac{1}{2}-x}{3}=\frac{x}{2}\left\{\frac{6}{x}-1\right\}$ to find the value of x.
Ans. $x=3$.

29. Given $3\cdot25x-5\cdot007-x=0\cdot2-0\cdot34x$ to find the value of x.
Ans. $x=2\cdot0\dot{1}0424\dot{7}$.

30. Given $\frac{7\cdot53x}{18}-100=\frac{2x}{5}-3\cdot86-\frac{x}{6}$ to find the value of x.
Ans. $x=+519\cdot\dot{6}7\dot{5}$.

31. Given $\frac{3abc}{a+b}+\frac{a^2b^2}{(a+b)^3}+\frac{(2a+b)b^2x}{a(a+b)^2}=3cx+\frac{bx}{a}$ to the value of x.
Ans. $x=\frac{ab}{a+b}$.

32. Given $\frac{cx^m}{a+bx}=\frac{fx^m}{d+ex}$ to find the value of x.
Ans. $x=\frac{af-cd}{ce-bf}$.

33. Given $\sqrt[3]{a^2+c}=\sqrt[4]{\frac{a^2+c}{d(x+g)}}$ to find the value of x.
Ans. $x=\frac{1}{d\sqrt[3]{a^2+c}}-g$.

34. Given $\sqrt[m]{a+x}=\sqrt[2m]{x^2+5ax+b^2}$ to find the value of x.
Ans. $x=\frac{a^2-b^2}{3a}$.

35. Given $\frac{bx}{2b-a}-\frac{(3bc+ad)x}{2ab(a+b)}-\frac{5ab}{3c-d}=\frac{(3bc-ad)x}{2ab(a-b)}-\frac{5a(2b-a)}{a^2-b^2}$ to find the value of x. *Ans.* $x=\frac{5a(2b-a)}{3c-d}$.

36. Given $\frac{7x^n}{x-1}=\frac{6x^{n+1}+x^n}{x+1}-\frac{3x^n+6x^{n+2}}{x^2-1}$ to find the value of x.

Ans. $x=-\frac{1}{12}$.

37. Given $\frac{9x+20}{36}=\frac{4x-12}{5x-4}+\frac{x}{4}$ to find the value of x.

Ans. $x=8$.

38. Given $\frac{7x+16}{21}-\frac{x+8}{4x-11}=\frac{x}{3}$ to find the value of x.

Ans. $x=8$.

39. Given $\frac{6x+7}{9}+\frac{7x-13}{6x+3}=\frac{2x+4}{3}$ to find the value of x.

Ans. $x=4$.

+ **40.** Given $\frac{5x+5}{x+2}-14=\frac{6x-12}{2x-2}-15$ to find the value of x.

Ans. $x=2$.

41. Given $\frac{x-2}{\sqrt{x}}=\frac{2\sqrt{x}}{3}$ to find the value of x. *Ans.* $x=6$.

42. Given $x+\sqrt{2ax+x^2}=a$ to find the value of x. *Ans.* $x=\frac{a}{4}$.

43. Given $2\sqrt{a^2+x^2}=4(a-\frac{1}{2}x)$ to find the value of x.

Ans. $x=\frac{3a}{4}$.

44. Given $a+x=\sqrt{a^2+x\sqrt{b^2+x^2}}$ to find the value of x.

Ans. $x=\frac{b^2-4a^2}{4a}$.

45. Given $\sqrt{3x-1}=2$ to find the value of x. *Ans.* $x=\frac{5}{3}$.

46. Given $\sqrt{x+x^2}=x+\frac{1}{3}$ to find the value of x. *Ans.* $x=\frac{1}{3}$.

47. Given $\sqrt[3]{3x+13}-4=0$ to find the value of x. *Ans.* $x=17$.

48. Given $\frac{\sqrt{a^2-x^2}}{\sqrt{a-x}}+x=a+2x$ to find the value of x.

Ans. $x=1-a$.

49. Given $\sqrt{4+\sqrt{x^4-x^2}}=x-2$ to find the value of x.

Ans. $x=2\frac{1}{8}$.

50. Given $(2+x)^{\frac{1}{2}}+x^{\frac{1}{2}}=4(2+x)^{-\frac{1}{2}}$ to find the value of x.

Ans. $x=\frac{2}{3}$.

51. Given $3\sqrt{2x+6}+3=15$ to find the value of x. *Ans.* $x=5$.

52. Given $\sqrt{x}+3=\sqrt{21+x}$ to find the value of x. *Ans.* $x=4$.

53. Given $x+\sqrt{a-x}=\dfrac{a}{\sqrt{a-x}}$ to find the value of x.

Ans. $x=a-1$.

54. Given $\dfrac{3x+1}{3x}-\dfrac{3(x-1)}{3x+2}=\dfrac{9}{11x}$ to find the value of x.

Ans. $x=\frac{32}{117}$.

55. Given $x+\dfrac{a}{b}x+\dfrac{c}{b}x=m$ to find the value of x.

Ans. $x=\dfrac{bm}{a+b+c}$.

56. Given $\sqrt{a+x}+\sqrt{a-x}=\sqrt{ax}$ to find the value of x.

Ans. $x=\dfrac{4a^2}{a^2+4}$.

57. Given $\sqrt{\dfrac{b}{a+x}}+\sqrt{\dfrac{c}{a-x}}=\sqrt[4]{\dfrac{4bc}{a^2-x^2}}$ to find the value of x.

Ans. $x=a\left(\dfrac{b+c}{b-c}\right)$.

58. Given $\sqrt{x+a}=c-\sqrt{x+b}$ to find the value of x.

Ans. $x=\left(\dfrac{c^2+b-a}{2c}\right)^2-b$.

59. Given $\sqrt{a}+\sqrt{x}=\sqrt{ax}$ to find the value of x.

Ans. $x=\dfrac{a}{(\sqrt{a}-1)^2}$.

60. Given $\sqrt{x-16}=8-\sqrt{x}$ to find the value of x. *Ans.* $x=25$.

61. Given $\sqrt{x+40}=10-\sqrt{x}$ to find the value of x. *Ans.* $x=9$.

62. Given $\sqrt{x-24}=\sqrt{x}-2$ to find the value of x. *Ans.* $x=49$.

63. Given $\sqrt{5}\times\sqrt{x+2}=\sqrt{5x}+2$ to find the value of x.

Ans. $x=\frac{9}{20}$.

64. Given $ax+a\sqrt{2ax+x^2}=ab$ to find the value of x.

Ans. $x=\dfrac{b^2}{2(a+b)}$.

65. Given $\dfrac{\sqrt{x}+2a}{\sqrt{x}+b}=\dfrac{\sqrt{x}+4a}{\sqrt{x}+3b}$ to find the value of x.

Ans. $x=\left(\dfrac{ab}{a-b}\right)^2$.

66. Given $\sqrt{4a+x}=2\sqrt{b+x}-\sqrt{x}$ to find the value of x.

Ans. $x=\dfrac{(b-a)^2}{2a-b}$.

67. Given $\dfrac{x-ax}{\sqrt{x}}=\dfrac{\sqrt{x}}{x}$ to find the value of x. *Ans.* $x=\dfrac{1}{1-a}$.

68. Given $\dfrac{\sqrt{ax}-b}{\sqrt{ax}+b}=\dfrac{3\sqrt{ax}-2b}{3\sqrt{ax}+5b}$ to find the value of x.

Ans. $x=\dfrac{9b^2}{a}$.

69. Given $\dfrac{\sqrt{4x+1}+2\sqrt{x}}{\sqrt{4x+1}-2\sqrt{x}}=9$ to find the value of x. *Ans.* $x=\frac{4}{9}$.

70. Given $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=b$ to find the value of x.

Ans. $x=\dfrac{2ab}{b^2+1}$.

71. Given $\dfrac{x+1}{x-1}\sqrt{\dfrac{x-1}{x+1}}=2$ to find the value of x. *Ans.* $x=1\frac{2}{3}$.

72. Given $\dfrac{\sqrt{6x}-2}{\sqrt{6x}+2}=\dfrac{4\sqrt{6x}-9}{4\sqrt{6x}+6}$ to find the value of x. *Ans.* $x=6$.

73. Given $a+b\sqrt[m]{x+d}=c$ to find the value of x.

Ans. $x=\left(\dfrac{c-a}{b}\right)^m-d$.

74. Given $\sqrt{x}+\sqrt{x-9}=\dfrac{36}{\sqrt{x-9}}$ to find the value of x.

Ans. $x=25$.

75. Given $\sqrt{1+x\sqrt{x^2+12}}=1+x$ to find the value of x.

Ans. $x=2$.

76. Given $\sqrt{x+\sqrt{x}}-\sqrt{x-\sqrt{x}}=\dfrac{3}{2}\left(\dfrac{x}{x+\sqrt{x}}\right)^{\frac{1}{2}}$ to find the value of x.

Ans. $x=\dfrac{25}{16}$.

77. Given $\frac{10+x}{5}:\frac{4x-9}{7}::14:5$ to find the value of x.

Ans. $x=4$.

78. Given $5x+\frac{7x+9}{4x+3}=9+\frac{10x^2-18}{2x+3}$ to find the value of x.

Ans. $x=3$.

79. Given $\sqrt[5]{9x-4}+6=8$ to find the value of x. *Ans.* $x=4$.

80. Given $\frac{17-4x}{4}:\frac{15+2x}{3}-2x::5:4$ to find the value of x.

Ans. $x=3$.

81. Given $16x+5:\frac{4x+14}{9x+31}::36x+10:1$ to find the value of x.

Ans. $x=5$.

82. Given $\frac{\sqrt{x}+\sqrt{b}}{\sqrt{x}-\sqrt{b}}=\frac{a}{b}$ to find the value of x. *Ans.* $x=b\left(\frac{a+b}{a-b}\right)^2$.

83. Given $\frac{\sqrt{9x}-4}{\sqrt{x}+2}=\frac{15+\sqrt{9x}}{\sqrt{x}+40}$ to find the value of x. *Ans.* $x=4$.

84. Given $a\sqrt[3]{bx-c}=d\sqrt[3]{ex+fx-g}$ to find the value of x.

Ans. $x=\frac{a^3c-d^3g}{a^3b-d^3(e+f)}$.

85. Given $\frac{1}{3}-\frac{7x-1}{6\frac{1}{2}-3x}=\frac{8}{3}\cdot\frac{x-\frac{1}{2}}{x-2}$ to find the value of x.

Ans. $x=\frac{14}{13}$.

86. Given $\sqrt{x}+\sqrt{x-9}=\frac{36}{\sqrt{x-9}}$ to find the value of x.

Ans. $x=25$.

87. Given $\frac{3x}{2}-\frac{81x^2-9}{(3x-1)(x+3)}=3x-\frac{3}{2}\cdot\frac{2x^2-1}{x+3}-\frac{57-3x}{2}$ to find the value of x. *Ans.* $x=10$.

88. Given $\frac{2}{19}\left\{\sqrt{x^2+39x+374}-\sqrt{x^2+20x+51}\right\}=\sqrt{\frac{x+22}{x+17}}$ to find the value of x. *Ans.* $x=78$.

89. Given $x^2+2x=(x+a)^2$ to find the value of x.

Ans. $x=\frac{a^2}{2(1-a)}$.

90. Given $x^3+2x^2+x=(x^2+3x)(x-1)+16$ to find the value of x.

Ans. $x=4$.

***91.** Given $\frac{x-5}{4}:x-5::\frac{2}{3}:\frac{3}{4}$ to find the value of x. *Ans.* $x=5$.

92. Given $\frac{(a+b)(x-b)}{a-b}-3a=\frac{4ab-b^2}{a+b}-2x+\frac{a^2-bx}{b}$ to find the value of x. *Ans.* $x=\frac{a^4+3a^3b+4a^2b^2-6ab^3+2b^4}{2b[2a^2+(a-b)b]}$.

93. Given $\frac{a}{bx}+\frac{c}{dx}+\frac{e}{fx}+\frac{g}{hx}=k$ to find the value of x.

Ans. $x=\frac{adfh+bcfh+bdeh+bdfg}{bdfhk}$.

94. Given $\frac{10x+17}{18}-\frac{12x+2}{13x-16}=\frac{5x-4}{9}$ to find the value of x.

Ans. $x=4$.

95. Given $\frac{2x+8\frac{1}{2}}{9}-\frac{13x-2}{17x-32}+\frac{x}{3}=\frac{7x}{12}-\frac{x+16}{36}$ to find the value of x. *Ans.* $x=4$.

96. Given $\frac{7x+6}{28}-\frac{2x+4\frac{2}{7}}{23x-6}+\frac{x}{4}=\frac{11x}{21}-\frac{x-3}{42}$ to find the value of x.

Ans. $x=4$.

97. Given $\frac{6-5x}{15}-\frac{7-2x^2}{14(x-1)}=\frac{1+3x}{21}-\frac{2x-\frac{11}{5}}{6}+\frac{1}{105}$ to find the value of x. *Ans.* $x=4$.

98. Given $\frac{a^2x}{b-c}-dc=bx-ac$ to find the value of x.

Ans. $x=\frac{c(b-c)(d-a)}{a^2-b^2+bc}$.

99. Given $\frac{18x-19}{28}+\frac{11x+21}{6x+14}=\frac{9x+15}{14}$ to find the value of x.

Ans. $x=7$.

100. Given $\frac{x}{2}-\frac{\frac{2x-3}{3}-\frac{3x-1}{4}}{\frac{x-1}{2}}=\frac{3}{2}\cdot\frac{x^2+2}{3x-2}$ to find the value of x.

Ans. $x=\frac{13}{3}$.

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* There is a peculiarity in this example. Any quantities whatever, whether in the ratio of $\frac{2}{3}$ to $\frac{3}{4}$ or not, when substitued for $\frac{2}{3}$ and $\frac{3}{4}$, will give the same answer. Can the student explain the reason?

QUESTIONS INVOLVING SIMPLE EQUATIONS CONTAINING ONE UNKNOWN QUANTITY.

QUESTION

(266.) 1. What number is that, the double of which exceeds its half by 6?

SOLUTION.

Let $x=$ the number.

Then, by the conditions of the question, we must have the equation

$$2x-\frac{x}{2}=6,$$
$$4x-x=12,$$
$$3x=12,$$
$$x=4, \text{ the number required.}$$

Another Solution.

Let $2x$ = the number.

Then, by the conditions of the question, we must have the equation

$$4x-x=6,$$
$$3x=6,$$
$$x=2,$$
$$2x=4, \text{ the number required.}$$

QUESTION

2. A person employed 4 workmen; to the first of whom he gave 2 dollars more than to the second; to the second, 3 dollars more than to the third; and to the third, 4 dollars more than to the fourth. Their wages amounted to 32 dollars. How much did each receive?

SOLUTION.

Let $x=$	the sum	received	by the	fourth,
then $x+4=$	"	"	"	third,
$x+7=$	"	"	"	second,
and $x+9=$	"	"	"	first.

By the conditions of the question, the sum of these must equal 32 dollars. Therefore,

$$4x+20=32,$$
$$4x=12,$$
$x=3$, the sum received by the fourth,
$x+4=7$, " " " third,
$x+7=10$, " " " second,
and $x+9=12$, " " " first.

QUESTION

3. What two numbers are to each other as 2 to 3; to each of which if 4 be added, the sums will be as 5 to 7?

SOLUTION.

Let $2x$ and $3x$ be the numbers.
Then, by the conditions of the question,

$$2x+4 : 3x+4 :: 5 : 7.$$

Whence, $14x+28=15x+20$,
$8=x$,
$2x=16$, the first number.
$3x=24$, the second number.

QUESTION

4. A person being asked the hour, answered that it was between five and six; and the hour and minute hands were together. What was the time?

SOLUTION.

Let $x=$ the time past 5.
Then, since the minute-hand goes 12 times as fast as the hour-hand, it follows, that $5+x$ is 12 times x,

$\therefore\ 12x=5+x$
$11x=5$
$x=\frac{5}{11}$ of an hour $=27$ minutes $16\frac{4}{11}$ seconds, the time past 5 o'clock.

QUESTION

5. What number is that to which if 1, 5, and 13, be severally added, the first sum shall be to the second as the second to the third?

SOLUTION.

Let $x=$ the number required.

Then, by the conditions of the question,

$$x+1 : x+5 :: x+5 : x+13.$$

Whose solution gives $x=3$.

QUESTION

6. A shepherd, in time of war, was plundered by a party of soldiers, who took $\frac{1}{4}$ of his flock, and $\frac{1}{4}$ of a sheep; another party took from him $\frac{1}{3}$ of what he had left, and $\frac{1}{3}$ of a sheep; then a third party took $\frac{1}{2}$ of what now remained, and $\frac{1}{2}$ of a sheep. After which he had but 25 sheep left. How many had he at first?

SOLUTION.

Let $x=$ the number he had at first.

Then $\frac{x}{4}+\frac{1}{4}=$ the number the first party took away.

Which, being subtracted from x, gives

$$\frac{3x}{4}-\frac{1}{4}= \text{ the number remaining.}$$

The second party took away $\frac{1}{3}$ of these, $+\frac{1}{3}$ of a sheep, which left

$$\frac{2}{3}\left(\frac{3x}{4}-\frac{1}{4}\right)-\frac{1}{3}, \text{ or } \frac{x}{2}-\frac{1}{2}.$$

Of these the third party took one-half $+\frac{1}{2}$ of a sheep, which left

$$\frac{1}{2}\left(\frac{x}{2}-\frac{1}{2}\right)-\frac{1}{2}, \text{ or } \frac{x}{4}-\frac{3}{4}.$$

$$\therefore \quad \frac{x}{4}-\frac{3}{4}= 25$$

$$x-3=100$$

$$x=103.$$

QUESTION

7. A man and his wife usually drank a vessel of beer in 12 days; but when the man was gone, it lasted the woman 30 days. In how many days would the man alone drink it?

SOLUTION.

Let x = the number of days it would take the man alone to drink it.

Then, the man, in 1 day, would drink $\frac{1}{x}$ of it.

The woman, " " " $\frac{1}{30}$ "

The man and woman together " $\frac{1}{12}$ "

$$\therefore\ \frac{1}{12}=\frac{1}{x}+\frac{1}{30}$$

$$5x=60+2x$$

$$3x=60$$

$x=20$, the time it would take the man alone to drink it.

QUESTION

8. A person engaged to reap a field of 35 acres, consisting partly of wheat and partly of rye. For every acre of rye he received 5 shillings; and what he received for an acre of wheat, augmented by one shilling, is, to what he received for an acre of rye, as 7 to 3. For his whole labor he received 260 shillings. What was the number of acres of each sort?

SOLUTION.

Let $x=$ the number of acres of wheat;

Then $35-x=$ the number of acres of rye;

Then $175-5x=$ the price of reaping the rye.

By the question, $3:7::5:1+$, the price of reaping an acre of wheat.

But $3:7::5:11\frac{2}{3}$.

Therefore, the price of reaping 1 acre of wheat is $10\frac{2}{3}=\frac{32}{3}$ shillings,

And " " " x acres " " $\frac{32x}{3}$ shillings.

$$\therefore\ \frac{32x}{3}+175-5x=260,$$

$$32x+525-15x=780,$$

$$17x=255,$$

$x=15$, the No. of acres of wheat.

$35-x=20$, " " rye.

QUESTION

9. The hold of a ship contained 442 gallons of water. This was emptied out by two buckets: the greater of which, holding twice as much as the other, was emptied twice in 3 minutes; but the less, three times in 2 minutes; and the whole time of emptying was 12 minutes. How much did each hold?

SOLUTION.

Let x = the number of gallons the less held.

Then $2x$=the number of gallons the greater held.

$4x=$ the gallons thrown out by the greater in 3 minutes
$16x=$ " " " " 12 "
$18x=$ " " " less " 12 "
$\therefore\ 16x+18x=34x=442,$
$x=13$, the No. of gallons the first bucket held.
$2x=26$, " " second " "

QUESTIONS.

1. What number is that, from the treble of which if 18 be subtracted, the remainder is 6? *Ans.* 8.

2. What number is that, the double of which exceeds $\frac{4}{5}$ of its half by 40? *Ans.* 25.

3. In fencing the side of a field, whose length was 450 yards, two workmen were employed; one of whom fenced 9 yards, and the other 6, per day. How many days did they work? *Ans.* 30.

4. A farmer sold 13 bushels of barley, at a certain price; and afterward 17 bushels, at the same rate; and at the second time received 36 dimes more than at the first. What was the price of a bushel? *Ans.* 90 cents.

5. A draper sold two pieces of cloth, by one of which he lost \$6 more than by the other: and his whole loss was \$5 less than treble the less loss. What were the losses sustained by each piece? *Ans.* \$11, and \$17.

6. A company settling their reckoning at a tavern, pay \$8 each; but observe, that if there had been 4 more, they should only have paid \$7 each? How many were there? *Ans.* 28.

7. Two workmen received the same sum for their labor; but if one had received \$15 more, and the other \$9 less, then one would have had just three times as much as the other. What did they each receive? *Ans.* \$21 each.

8. What number is that, the treble of which is as much above 40, as its half is below 51? *Ans.* 26.

9. A person has a certain number of horses at a livery stable, and 3 times as many at grass. He keeps 15 in constant employment; and his whole number is 7 times the number in the stable. What was the whole number? *Ans.* 35.

10. Two men at the distance of 150 miles set out to meet each other; one goes 3 miles while the other goes 7. How much of the distance does each travel? *Ans.* One 45, and the other 105 miles.

11. A person put out a certain sum at interest for $6\frac{1}{2}$ years, at 5 per cent. simple interest; and found that if had put out the same sum for 12 years and 9 months, at 4 per cent., he would have received \$185 more. What was the sum put at interest. *Ans.* \$1000.

12. From two casks of equal size are drawn quantities, which are in the proportion of 6 to 7; and it appears that if 16 gallons less had been drawn from that which is now the emptier, only half as much would have been drawn from it as from the other. How many gallons were drawn from each?

Ans. 24 gallons from one, and 28 gallons from the other.

13. Out of a certain sum, a man paid his creditors \$96; half of the remainder he lent his friend; he then spent $\frac{1}{5}$ of what now remained; and after all these deductions had $\frac{1}{10}$ of his money left. How much had he at first? *Ans.* \$128.

14. Six hundred persons voted upon a disputed question, which was lost by a certain number. The same number of persons having voted again upon the same question, it was from some change in circumstances carried by twice as many as it was before lost by; and the new majority was to the former one as 8 : 7. How many changed their minds? *Ans.* 150.

15. A sportsman, keeping an account of the number of birds he killed, found that each succeeding season he wanted 50, in order that the number killed might bear the proportion of 3 : 2 to the number killed in the preceding year. In the fourth year he found that he had killed 170 fewer than three times the number killed in the first year. How many did he kill the first year? *Ans.* 180.

16. Several detachments of artillery divided a certain number of cannon balls. The first took 72, and $\frac{1}{9}$ of the remainder; the next 144, and $\frac{1}{9}$ of the remainder; the third 216, and $\frac{1}{9}$ of the remainder; the fourth 288, and $\frac{1}{9}$ of those that were left; and so on; when it was found that the balls had been equally divided. What was the number of balls and detachments?

Ans. 4608 balls, and 8 detachments.

17. Two persons, *A* and *B*, start at the same time for a race which lasted 6 minutes. Now after galloping 4 minutes at the same uniform pace at which each started, the distance between them is $\frac{1}{440}$ the part of the whole length of the course. They continue to run for 1 minute more, at the same speed as at first; and then *B*, who is last,

quickens the speed of his horse 20 yards a minute, and comes in exactly two yards before A, whose horse had run at the same uniform pace throughout. What was the length of the course?

Ans. 3 miles.

18. A packet sailing from Dover with a fair wind, arrives at Calais in 2 hours; and on its return the wind being contrary, it proceeds 6 miles an hour slower than it went. Now when it is half way over, the wind changing, it sails 2 miles an hour faster, and reaches Dover sooner than it would have done had the wind not changed, in the proportion of 6 : 7. What were the rates of sailing and the distance between Dover and Calais?

Ans. On its return it sails 5 and 7 miles an hour, and the distance is 22 miles.

19. A farmer has a stack of hay, from which he sells a quantity which is to the quantity remaining in the proportion of 4 to 5. He then uses 15 loads, and finds that he has a quantity left which is to the quantity sold as 1 to 2. How many loads did the stack contain at first? *Ans.* 45.

20. In a naval engagement, the number of ships taken was 7 more, and the number burnt 2 fewer than the number sunk. Fifteen escaped, and the fleet consisted of 8 times the number sunk. Of how many ships did the fleet consist? *Ans.* 32.

21. At the review of an army, the troops were drawn up in a solid mass, 40 deep, when there were just $\frac{1}{4}$ as many men in front as there were spectators. Had the depth, however, been increased by 5, and the spectators been drawn up in the mass with the army, the number of men in front would have been 100 fewer than before. Of what number did the army consist? *Ans.* 180000.

22. A and B playing at billiards, A bet 5 dollars to 4 on every game, and found that after a certain amount of games, he had won 10 dollars. Had B won one game more, the number won by him would have been to the number won by A as 3 to 4. How many did each win? *Ans.* A won 20, and B 14.

23. A besieged garrison had such a quantity of bread as would, if distributed to each at 10 ounces a day, last 6 weeks, but having lost 1200 men in a sally, the governor was enabled to increase the allowance to 12 ounces per day for 8 weeks. What was the number of men at first in the garrison? *Ans* 3200.

24. During a panic, there was a run on two bankers A and B. B stopped payment at the end of 3 days, in consequence of which the alarm increased, and the daily demand for cash on A being trebled, A failed at the end of 2 more days. But if A and B had joined their capitals, they might both have stood the run, as it was at first, for 7 days, at the end of which time B would have been indebted to A \$4000. What was the daily demand for cash at A's bank at first?

Ans. \$2000.

25. There are two numbers in the proportion of $\frac{1}{2}$ to $\frac{2}{3}$, which being increased respectively by 6 and 5, are in the proportion of $\frac{2}{3}$ to $\frac{1}{2}$. What are the numbers? *Ans.* 30 and 40.

26. A gentleman meeting 4 poor persons, distributed 60 cents among them: to the second he gave twice, to the third thrice, and to the fourth four times as much as to the first. How much did he give to each? *Ans.* 6, 12, 18, and 24 cents respectively.

27. A farmer has two flocks of sheep, each containing the same number. From one of these he sells 39, and from the other 93; and finds just twice as many remaining in one as in the other. How many did each flock originally contain? *Ans.* 147.

28. Four places are situated in the order of the four letters A, B, C, D. The distance from A to D is 34 miles; the distance from A to B : distance from C to D : : 2 : 3, and $\frac{1}{4}$ of the distance from A to B added to half the distance from C to D is 3 times the distance from B to C. What are the respective distances?

Ans. $AB=12$, $BC=4$, and $CD=18$.

29. In a mixture of wine and cider, half of the whole $+25$ gallons was wine, and $\frac{1}{3}$ of the whole -5 gallons was cider. How many gallons were there of each?

Ans. 85 gallons of wine, and 35 of cider.

30. A footman who contracted for £8 a year, and a livery suit, was turned away at the end of 7 months, and received only £2 3*s.* 4*d.* and his livery. What was its value? *Ans.* £6.

31. A cistern into which water was let by two pipes, A and B, will be filled by them both running together in 12 hours, and by the pipe A, alone, in 20 hours. In what time will it be filled by the pipe B alone? *Ans.* 30 hours.

32. A person has two sorts of wine, one worth 20 pence a quart,

and the other 12 pence; from which he would mix a quart to be worth 14 pence. How much of each must he take?

Ans. He must take $\frac{1}{4}$ of the first, and $\frac{3}{4}$ of the second.

33. A hare, 50 of her leaps before a greyhound, takes 4 leaps to the greyhound's 3; but 2 of the greyhound's leaps are as much as 3 of the hare's. How many leaps must the greyhound take to catch the hare? *Ans.* 300.

34. Two pieces of cloth of equal goodness, but of different lengths, were bought, the one for \$5, and the other for \$6$\frac{1}{2}$. Now, if the lengths of both pieces were increased by 10, the numbers resulting would be in the proportion of 5 to 6. How long was each piece, and what was the cost a yard?

Ans. One piece was 20 yards long, and the other 26. Cost, 25 cts. a yard.

35. Two persons, *A* and *B*, have both the same annual income. *A* lays by $\frac{1}{5}$ of his: but *B*, by spending \$80 per annum more than *A*, at the end of 4 years finds himself \$220 in debt. How much did each receive and spend annually?

Ans. The annual income of each is \$125, and *A*'s annual expenditure is \$100, and *B*'s \$180.

36. An egg-merchant meeting with three customers, sells to the first of them half his stock and 1 egg more; to the second he disposes of half the remainder and 2 eggs more; and to the third half of what he then had left and 3 eggs more; and afterward discovers that he has parted with his whole stock. What number had he at first? *Ans.* 34.

37. A person disposes of turkeys at as many dimes each as the number he has, and returning 1 dime finds that if he had had one more to sell on the same condition, and had returned 2 dimes, he would have received 20 dimes more from his bargain. What number did he dispose of? *Ans.* 10.

38. A gentleman bequeaths his property as follows: To his eldest child he leaves \$1800, and $\frac{1}{6}$ of the rest of his property; to the second, twice that sum and $\frac{1}{6}$ of what then remained; to the third, three times the same sum and $\frac{1}{6}$ of the remainder, and so on; and by this arrangement his property is divided equally among his children. How many children were there, and what was the fortune of each?

Ans. 5, and \$9000, the fortune of each.

39. A and B possess certain sums of money, such that if they gain \$$a$ and \$$b$ respectively, A will be m times as rich as B; but if they gain \$$c$ and \$$d$ respectively, A becomes possessed of n times as much as B. How much money has each?

$$\textit{Ans.}\quad \begin{cases} \dfrac{m(nd-c)-n(mb-a)}{m-n} = A\text{'s money at first.} \\[2ex] \dfrac{(nd-c)-(mb-a)}{m-n} = B\text{'s money at first.} \end{cases}$$

40. The crew of a ship consisted of her complement of sailors and a number of soldiers. Now there were 22 sailors to every 3 guns and 10 more. Also, the whole number of persons was 5 times the number of soldiers and guns together. But after an engagement, in which the slain were $\frac{1}{4}$ of the survivors, there wanted 5 to be 13 men to every 2 guns. What was the number of guns, soldiers, and sailors?

Ans. 90 guns, 670 sailors, and 55 soldiers.

41. An express set out to travel 140 miles in 4 days, but in conse-
40 quence of the badness of the roads, he found that he must go 5 miles the second day, 9 the third, and 14 the fourth day, less than the first. How many miles did he travel each day? 47, 37, 33, 28 (140 (24

Ans. 67, 62, 58, and 53 miles.

42. The estate of a bankrupt, valued at \$21000, is to be divided among four creditors proportionably to what is due them. The debts due to A and B are as $2:3$; B's and C's claims are in the ratio of $4:5$; and C's, and D's in the ratio of $6:7$. What sum must each receive? *Ans.* A \$3200, B \$4800, C \$6000, and D \$7000.

43. There are two towns, A and B, which are 131 miles distant from each other. A coach sets out from A at six o'clock in the morning, and travels at the rate of 4 miles an hour without intermission, in the direct road toward B. At 2 o'clock in the afternoon of the same day, a coach sets out from B to go to A, and goes at the rate of 5 miles an hour constantly. Where will they meet?

Ans. 76 miles from A, and 55 from B.

44. A waterman finds by experience that he can with the advantage of the common tide row down a river from A to B, which is 18 miles, in 1 hour and a half, and that to return from B to A against an equal tide, though he rows back along the shore, where the stream is only $\frac{3}{5}$ as strong as in the middle, takes him just 2 hours and a quarter. At what rate does the tide run per hour in the middle, where it is the strongest? *Ans.* $2\frac{1}{2}$ miles per hour.

45. The ingredients of a loaf of bread are rice, flour, and water, and the weight of the whole is 15 lbs. The weight of the rice augmented by 5 lbs. is $\frac{2}{3}$ of the weight of the flour, and the weight of the water is $\frac{1}{5}$ of the weight of the flour and the rice together. What is the weight of each? *Ans.* Rice 2 lbs., flour $10\frac{1}{2}$ lbs., water $2\frac{1}{2}$ lbs.

46. Suppose two hands of a watch, (a) and (b), were together on Sunday noon, and the motion of each was such that (a) moved round the horary circle in one hour, and (b) in $1\frac{1}{60}$ hour. When will they be together again for the first time? *Ans.* 61 hours.

47. The rent of an estate this year is greater by 8 per cent. than it was last year. This year's rent is $1890. What was the rent of last year? *Ans.* $1750.

48. A merchant increases his capital yearly by 20 per cent., and takes from it every year $1000 for the support of himself and family. After he had carried on his business, in this manner, for three years, he finds, after deducting the usual sum, $1000, that his capital has increased $200 more than $\frac{3}{5}$ of the original sum. What was the original capital? *Ans.* $30000.

49. A person has 4 wine casks of different sizes. When he fills the 2d empty cask from the first full one, there remains in the first only $\frac{4}{7}$ of the wine; when he fills the 3d empty cask from the 2d full one, there is left in the 2d only $\frac{1}{4}$ of the wine; but when he attempts to fill the 4th empty cask from the 3d full one, then only $\frac{9}{16}$ of the 4th is filled, and if he wished to fill the 3d and 4th empty casks from the first full one, then these would not only be filled, but he would have 15 gallons remaining. How many gallons does each of these four casks contain?

Ans. The 1st, 140; the 2d, 60; the 3d, 45; and the 4th, 80 gallons.

50. A father leaves a number of children, and a certain sum of money, which they are to divide among them as follows: The first is to receive $100, and a 10th part of the remainder; after this, the second receives $200, and a 10th part of the residue; again, the third receives $300, and a 10th part of the remainder; and so on. At last it is found that all the children have received the same sum. What was the fortune left, and how many children were there?

Ans. $8100, and 9 children.

51. What number is that which, if it be increased by 7, the square

root of the sum shall be equal to the square root of the number itself and 1 more? *Ans.* 9.

52. A person wishes to give 3 cents a-piece to some beggars, but finds he has not money enough by 8 cents; but if he gives them 2 cents a-piece, he will have 3 cents remaining. What is the number of beggars? *Ans.* 11.

53. What are the two parts of 60, such that their product is equal to three times the square of the less? *Ans.* 15 and 45.

54. In the composition of a quantity of gunpowder, the nitre was 10 lbs. more than $\frac{2}{3}$ of the whole, the sulphur $4\frac{1}{2}$ lbs. less than $\frac{1}{6}$ of the whole, and the charcoal 2 lbs. less than $\frac{1}{7}$ of the nitre. What was the amount of gunpowder? *Ans.* 69 lbs.

55. A person engaged to work a days on these conditions: For each day he worked he was to receive b cents, for each day he was idle he was to forfeit c cents. At the end of a days he received d cents. How many days was he idle?

Ans. $\dfrac{ab-d}{b+c}$ days.

56. What must the fortune and number of children be, when, in general, the first receives a dollars, together with the nth part of the remainder; and each succeeding child a dollars more, together with the nth part of the remainder, and it is found, at last, that they have all received the same sum?

Ans. The fortune, $=(n-1)^2a$, and children, $=n-1$.

57. A person wishes to make the following payments at 4 different periods; one sum a in l, a sum b in m, a sum c in n, and a sum d in p months. If he wishes to pay his whole debt, $a+b+c+d$, at once, at what period must he do it?

Ans. $\dfrac{al+bm+cn+dp}{a+b+c+d}$ months.

58. A master mason has engaged a number of masons for the erection of a building. He finds, after entering into a calculation, that if he gave each man m shillings a day, he would daily expend a shillings less than was assigned for that purpose by the estimate, and that he would lose b shillings, if he should give each n shillings. How many men did he hire, and what was the daily wages of each?

Ans. $\left\{\begin{array}{l}\text{The number of masons was } \dfrac{a+b}{n-m}. \\ \text{The daily wages of each was } \dfrac{an+bm}{a+b} \text{ shillings.}\end{array}\right.$

59. What number must be added to each of the two given numbers, a and b, that the sums may be as $m:n$?

Ans. $\dfrac{mb-na}{n-m}$.

60. Three masons are employed in building a wall. The first builds 8 cubic feet in 5 days; the 2d, 9 cubic feet in 4 days; and the 3d, 10 cubic feet in 6 days. How much time will these masons need, when they work together, to build 756 cubic feet of the wall?

Ans. $137\frac{13}{331}$ days.

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE, CONTAINING TWO UNKNOWN QUANTITIES.

(267.) Simultaneous Equations are such as must exist at the same time, the values of the unknown quantities in each equation being restricted by the other.

PROBLEM.

(268.) To discuss the nature of simultaneous equations.

DISCUSSION.

Let $x+y=6$. This equation can be satisfied by the following positive integral values for x and y.

$$\begin{array}{rcl} x=1 & \text{and} & y=5, \\ x=2 & \text{"} & y=4, \\ x=3 & \text{"} & y=3, \\ x=4 & \text{"} & y=2, \\ \text{or, } x=5 & \text{"} & y=1. \end{array}$$

Also, if $x=0$, y must $=6$, and if $x=6$, y must $=0$. We might assign negative values for x, and the corresponding values of y would be obtained by subtracting the value of x from 6. Thus, $x=-2$, $y=6-(-2)=8$.

We can also assign, for the value of x, any proper or improper fraction, and the corresponding value of y must be this value sub-

tracted from 6. There are, then, an infinite number of values which may be assigned to x and y which will satisfy the given equation.

Suppose, now, we have $3x+2y=14$. In this equation, also, x and y may have an infinite number of values. But, if we wish $x+y=6$ and $3x+2y=14$ to exist at the same time, or, in other words, that each of these equations must be restricted by the other, x and y can only have those values which will at the same time satisfy both equations.

We have, then, this problem, *to find what values of x and y will satisfy $x+y=6$, provided* $3x+2y=14$.

If $x=2$ and $y=4$, the first equation will be satisfied, and these values will also be found to satisfy the second.

It will hereafter be found, that of all values of x and y which will satisfy the first equation, $x=2$ and $y=4$ are the only ones that will, at the same time, satisfy the second equation.

$x+y=6$ is called an *indeterminate equation* when it has no other equation to limit it. In like manner $x+y+z=9$ is an indeterminate equation, when it has no other equation to limit it. In general, *that equations may be determinate there must be as many equations as there are unknown quantities.*

It must be carefully observed that the equations must all be independent, that is, that no equation be the result of an *addition*, *subtraction*, *multiplication*, or *division* performed upon one of the others. Thus $x+y=6$, and $x+y+3=9$ are not independent equations. The same may be said of $x+y=6$, and $x+y-3=3$; $x+y=6$ and $2x+2y=12$; $x+y=6$ and $\frac{1}{3}x+\frac{1}{3}y=2$.

ELIMINATION.

(269.) ELIMINATION is the process of deducing from two or more simultaneous equations, a single equation containing one unknown quantity.

(270.) *Simultaneous equations* are of the first degree when the equation deduced from them is of the first degree, of is a simple equation.

ELIMINATION BY SUBSTITUTION.

(271.) *Elimination by substitution* is finding an expression for the value of an unknown quantity in one equation, and substituting this value for the same unknown quantity in another equation.

PROBLEM.

(272.) To eliminate by substitution from two simultaneous equations.

RULE.

Find an expression for the value of an unknown quantity in one equation, and substitute this expression for the same unknown quantity in the other equation, and there will result a single equation containing but one unknown quantity.

PROBLEM

(273.) 1. Given $\left\{\begin{array}{ll} ax+by=c & (1) \\ mx+ny=d & (2) \end{array}\right\}$ to find the values of x and y.

SOLUTION.

$y=\frac{c-ax}{b}$ (3) = value of y in (1).

$ny=\frac{nc-nax}{b}$ (4) = (3) $\times n$.

$mx+\frac{nc-nax}{b}=d$ (5) = $\left\{\begin{array}{l}\text{value of } ny \text{ in (4) sub-} \\ \text{stituted in (2).}\end{array}\right.$

$bmx+nc-nax=bd.$

$bmx-nax=bd-nc.$

$(bm-na)x=bd-nc.$

$x=\frac{bd-nc}{bm-na}$ (8).

$ax=\frac{abd-acn}{bm-na}$ (9) = (8) $\times a$.

$\frac{abd-nac}{bm-na}+by=c$ (10) = $\left\{\begin{array}{l}\text{value of } ax \text{ in (9)} \\ \text{substituted in (1).}\end{array}\right.$

$by=c-\frac{abd-nac}{bm-na}.$

$y=\frac{c}{b}-\frac{abd-nac}{b^2m-abn}.$

$y=\frac{cm-ad}{bm-na}.$

PROBLEM

2. Given $\left\{\begin{array}{l} 4x+\ \ 9y=51\ (1) \\ 8x-13y=\ \ 9\ (2) \end{array}\right\}$ to find the values of x and y.

SOLUTION.

$4x=51-9y$ (3)=(1) transposed.

$8x=102-18y$ (4)=(3)$\times$2.

$102-18y-13y=9$ (5)= $\left\{\text{value of } 8x \text{ in (4) substituted in (2).}\right.$

$-31y=-93.$ (6)=(5) transposed and added.

$y=3.$ (7)=(6)$\div-31$.

Since, $y=3$, we have instead of $4x=51-9y$ the equation

$4x=51-27=24$ (8)=value of $9y$ substituted in (3).

$x=6.$

PROBLEM

3. Given $\left\{\begin{array}{l} x+y\ =3\ (1) \\ x^2-y^2=3\ (2) \end{array}\right\}$ to find the values of x and y.

SOLUTION.

The second of these equations is not of the first degree, but the equation found by eliminating will be of the first degree. The second equation is in fact the product of $x-y=1$ by 3, for it may be put in the form $(x+y)(x-y)=3$, and we know that $x+y=3$ from the first equation.

We have then to find the values of x and y in the equations

$x+y=3$ (3).

$x-y=1$ (4).

$x=1+y$ (5)=(4) transposed.

$1+y+y=3$ (6)=value of x in (5) substituted in (3).

$2y=2.$

$y=1.$

$x=2.$

PROBLEM

4. Given $\left\{\begin{array}{l} \dfrac{2x-y}{2}+14=18\ (1) \\ \dfrac{2y+x}{3}+16=19\ (2) \end{array}\right\}$ to find the values of x and y.

SOLUTION.

$$2x-y=8 \qquad (3)=(1)\times 2 \text{ and terms transposed.}$$
$$2y+x=9 \qquad (4)=(2)\times 3 \text{ and terms transposed.}$$
$$x=9-2y \qquad (5)=(4) \text{ transposed.}$$
$$2x=18-4y \qquad (6)=(5)\times 2.$$
$$18-4y-y=8 \qquad (7)= \text{ value of } 2x \text{ in (6) substituted in (3).}$$
$$-5y=-10.$$
$$y=2.$$
$$x=9-2y.$$
$$x=9-4.$$
$$x=5.$$

EXAMPLES.

1. Given $\left\{\begin{array}{l}5x+8y=31,\\ 2x+2y=10,\end{array}\right\}$ to find the values of x and y.

Ans. $x=3$, $y=2$.

2. Given $\left\{\begin{array}{l}6x-9y=9,\\ 2x+7y=13,\end{array}\right\}$ to find the values of x and y.

Ans. $x=3$, $y=1$.

3. Given $\left\{\begin{array}{l}5x-3y=11,\\ 15x+2y=66,\end{array}\right\}$ to find the values of x and y.

Ans. $x=4$, $y=3$.

4. Given $\left\{\begin{array}{l}2x-y=1,\\ 7x+9y=16,\end{array}\right\}$ to find the values of x and y.

Ans. $x=1$, $y=1$.

5. Given $\left\{\begin{array}{l}5x-\dfrac{4y+7x}{6}=8,\\ 7x-\dfrac{4y}{11}+\dfrac{7x-2y}{6}=3y-8,\end{array}\right\}$ to find the values of x and y.

Ans. $x=4$, $y=11$.

6. Given $\left\{\begin{array}{l}3x-\dfrac{7y-2}{5}=7-\dfrac{3y+7}{4},\\ 5x-\dfrac{7y}{11}-5=7+\dfrac{y-7}{4},\end{array}\right\}$ to find the values of x and y.

Ans. $x=4$, $y=11$.

7. Given $\left\{\begin{array}{l}7x+11y=40,\\ 5x-2y=9,\end{array}\right\}$ to find the values of x and y.

Ans. $x=\dfrac{179}{69}$, $y=\dfrac{137}{69}$.

$10x-6y=22$

$11x = 44$ $x=4$

8. Given $\left\{\begin{array}{l}3x+5y=19,\\7x-2y=17,\end{array}\right\}$ to find the values of x and y.

Ans. $x=3$, $y=2$.

9. Given $\left\{\begin{array}{l}5x-3y=11,\\7x-2y=22,\end{array}\right\}$ to find the values of x and y.

Ans. $x=4$, $y=3$.

10. Given $\left\{\begin{array}{l}5x+4y=26,\\6x+4y=28,\end{array}\right\}$ to find the values of x and y.

Ans. $x=2$, $y=4$.

ELIMINATION BY COMPARISON.

(274.) Elimination by Comparison is finding an expression for the value of an unknown quantity in one equation, and also, an expression for the value of the same unknown quantity in another equation, and putting the expressions equal to each other.

PROBLEM.

(275.) To eliminate by comparison from two simultaneous equations.

RULE.

Find an expression for the value of one of the unknown quantities in the first equation, and put it equal to an expression for the value of the same unknown quantity found from the second equation.

PROBLEM

(276.) 1. Given $\left\{\begin{array}{l}ax+by=m\ (1),\\cx+dy=n\ (2),\end{array}\right\}$ to find the values of x and y.

SOLUTION.

$ax=m-by$ (3)=(1) transposed.

$x=\dfrac{m-by}{a}$ (4)=(3)$\div a$.

$cx=n-dy$ (5)=(2) transposed.

$x=\dfrac{n-dy}{c}$ (6)=(5)$\div c$.

$\dfrac{m-by}{a}=\dfrac{n-dy}{c}$ (7)= value of x in (4) and in (6) equated.

$$cm-bcy=an-ady$$
$$ady-bcy=an-cm$$
$$(ad-bc)y=an-cm$$
$$y=\frac{an-cm}{ad-bc}.$$

The value of x may be obtained by equating the expressions for the

value of y, or it may be obtained from either of the given equations by putting in it instead of y its value as found above. Proceeding by either of these methods, we would find $x=\frac{md-nb}{ad-bc}$.

PROBLEM

(277.) 2. Given $\left\{\begin{array}{l}11x+3y=100\ (1),\\ 4x-7y=\quad 4\ (2),\end{array}\right\}$ to find the values of x and y.

SOLUTION.

$$x=\frac{100-3y}{11}$$

$$x=\frac{4+7y}{4}$$

$$\frac{4+7y}{4}=\frac{100-3y}{11}$$

$$44+77y=400-12y$$

$$89y=356$$

$y=4$. Since, $x=\frac{4+7y}{4}$ and $y=4$,

we have $x=\frac{4+28}{4}=8$.

PROBLEM

3. Given $\left\{\begin{array}{l}7x+3\tfrac{1}{17}y=59\ (1),\\ 64x-3\tfrac{1}{17}y=12\ (2),\end{array}\right\}$ to find the values of x and y.

SOLUTION.

$3\tfrac{1}{17}y=59-\ 7x$ $\quad$ (3)=(1) transposed.

$-3\tfrac{1}{17}y=7x-\ 59$ $\quad$ (4)=(3) with signs changed.

$-3\tfrac{1}{17}y=12-64x$ $\quad$ (5)=(2) transposed.

$7x-59=12-64x$ $\quad$ (6)= value of $-3\tfrac{1}{17}y$ in (4) and in (5) equated

$x=1$. $\quad$ Since $3\tfrac{1}{17}y=59-7x$ and $x=1$, we have

$$3\tfrac{1}{17}y=59-7=52$$

$$52y=17\,.\,52$$

$$y=17$$

REMARK.—This solution shows that all that is necessary in eliminating by comparison, is to find an expression for the value of one of the unknown quantities when affected by a certain coefficient, and then find from the other equation an expression for the same, and then equate them.

1. Given $\left\{\begin{array}{l} x+y=a, \\ x-y=b, \end{array}\right\}$ to find the values of x and y.

Ans. $x=\dfrac{a+b}{2}$, $y=\dfrac{a-b}{2}$.

*2. Given $\left\{\begin{array}{l} 3x+2y=118, \\ x+5y=191, \end{array}\right\}$ to find the values of x and y.

Ans. $x=16$, $y=35$.

3. Given $\left\{\begin{array}{l} ax=by \\ x+y=c \end{array}\right\}$ to find the values of x and y.

Ans. $x=\dfrac{bc}{a+b}$, $y=\dfrac{ac}{a+b}$.

4. Given $\left\{\begin{array}{c} 7y=2x-3y, \\ 19x=60y+621\frac{1}{4}, \end{array}\right\}$ to find the values of x and y

Ans. $x=88\frac{3}{4}$, $y=17\frac{3}{4}$.

5. Given $\left\{\begin{array}{c} 13x+7y-341=7\frac{1}{2}y+43\frac{1}{2}x, \\ 2x+\frac{1}{2}y=1, \end{array}\right\}$ to find the values of x and y.

Ans. $x=-12$, $y=50$.

6. Given $\left\{\begin{array}{c} 5x-8\frac{1}{2}=7y-44, \\ 2x=y+\frac{5}{7}, \end{array}\right\}$ to find the values of x and y.

Ans. $x=4\frac{1}{2}$, $y=8\frac{2}{7}$.

7. Given $\left\{\begin{array}{l} x+15y=53, \\ y+3x=27, \end{array}\right\}$ to find the values of x and y.

Ans. $x=8$, $y=3$.

8. Given $\left\{\begin{array}{l} 4x+9y=51, \\ 8x-13y=9, \end{array}\right\}$ to find the values of x and y.

Ans. $x=6$, $y=3$.

9. Given $\left\{\begin{array}{l} \dfrac{7+x}{5}-\dfrac{2x-y}{4}=3y-5, \\ \dfrac{5y-7}{2}+\dfrac{4x-3}{6}=18-5x, \end{array}\right\}$ to find the values of x and y.

Ans. $x=3$, $y=2$.

10. Given $\left\{\begin{array}{l} \dfrac{3x-1}{5}+3y-4=15 \\ \dfrac{3y-5}{6}+2x-8=7\frac{2}{3} \end{array}\right\}$ to find the values of x and y.

Ans. $x=7$, $y=5$.

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* Multiply the 2d equation by 3, and equate the values of $3x$.

ELIMINATION BY ADDITION AND SUBTRACTION.

(278.) *Elimination by addition and subtraction* is multiplying or dividing two equations so as to make the coefficients of one of the unknown quantities the same in both equations, and then subtracting or adding these equations according as the signs of these terms are like or unlike.

PROBLEM.

(279.) To eliminate by addition and subtraction from two simultaneous equations.

RULE.

Multiply or divide, if necessary, in such a manner as to cause one of the unknown quantities to have the same coefficient in both equations; and then add these equations if the signs of these terms are unlike, or subtract one from the other if the signs are alike.

PROBLEM

(280.) 1. Given $\left\{\begin{array}{ll} ax+by=m & (1), \\ cx+dy=n & (2), \end{array}\right\}$ to find the values of x and y.

SOLUTION.

$$\begin{array}{ll} acx+bcy=cm & (3)=(1)\times c, \\ acx+ady=an & (4)=(2)\times a, \\ (bc-ad)y=cm-an & (5)=(3)-(4), \\ y=\dfrac{cm-an}{bc-ad} & (6)=(5)\div(bc-ad). \end{array}$$

The value of x may be obtained in the same way by multiplying the first equation by d, and the second by b, and subtracting, or by substituting in either of the given equations the value of y as already found. By adopting either of these modes, we would get $x=\dfrac{md-nb}{ad-bc}$.

REMARK.—We should always eliminate those terms which require the least preparation. In the example just given there is no preference, as the operation can not be shortened, because a and c, and b and d are prime to each other.

PROBLEM

2. Given $\left\{\begin{array}{ll} 2y-\dfrac{x+3}{4}=7+\dfrac{3x-2y}{5} & (1), \\ 4x-\dfrac{8-y}{3}=24\tfrac{1}{2}-\dfrac{2x+1}{2} & (2), \end{array}\right\}$ to find the values of x and y.

SOLUTION.

$$\begin{array}{rl} 48y-17x=155 & (3)=(1)\times 20, \&c., \\ 2y+30x=160 & (4)=(2)\times 6, \&c., \\ 48y+720x=3840 & (5)=(4)\times 24, \\ 737x=3685 & (6)=(5)-(3), \\ x=5 & (7)=(6)\div 737, \\ 2y=160-30x & (8)=(4) \text{ transposed.} \\ \therefore 2y=160-150=10, & \\ y=5. & \end{array}$$

PROBLEM

3. Given $\left\{\begin{array}{ll} 16x+10y=42 & (1), \\ 5x-3\tfrac{1}{3}y=6\tfrac{2}{3} & (2), \end{array}\right\}$ to find the values of x and y.

SOLUTION.

$$\begin{array}{rl} 15x-10y=20 & (3)=(2)\times 3, \\ 31x=62 & (4)=(1)+(3), \\ x=2 & (5)=(4)\div 31, \\ 10y=42-16x & (6)=(1) \text{ transposed,} \\ \therefore 10y=42-32=10, & \\ y=1. & \end{array}$$

EXAMPLES.

1. Given $\left\{\begin{array}{l} \dfrac{x}{6}+\dfrac{y}{4}=6, \\ \dfrac{x}{4}+\dfrac{y}{6}=5\tfrac{2}{3} \end{array}\right\}$ to find the values of x and y.

Ans. $x=12$, $y=16$.

2. Given $\left\{\begin{array}{l} 9x+\dfrac{8y}{5}=70, \\ 7y-\dfrac{13x}{3}=44 \end{array}\right\}$ to find the values of x and y.

Ans. $x=6$, $y=10$.

3. Given $\left\{\begin{array}{l} (x+5)(y+7)=(x+1)(y-9)+112, \\ 2x+10=3y+1 \end{array}\right\}$ to find the values of x and y.

Ans. $x=3$, $y=5$.

4. Given $\left\{\begin{array}{l}\dfrac{a}{b+y}=\dfrac{b}{3a+x}\\ ax+2by=c\end{array}\right\}$ to find the the values of x and y.

Ans. $x=\dfrac{2b^2-6a^2+c}{3a}$, $y=\dfrac{3a^2-b^2+c}{3b}$.

5. Given $\left\{\begin{array}{l}x+1-\dfrac{3y+4x}{7}=7-\dfrac{9y+33}{14},\\ y-3-\dfrac{5x-4y}{2}=x-\dfrac{11y-19}{4}\end{array}\right\}$ to find the values of x and y.

Ans. $x=6$, $y=5$.

6. Given $\left\{\begin{array}{l}\dfrac{3x+4y+3}{10}-\dfrac{2x+7-y}{15}=5+\dfrac{y-8}{5},\\ \dfrac{9y+5x-8}{12}-\dfrac{x+y}{4}=\dfrac{7x+6}{11},\end{array}\right\}$ to find the values of x and y.

Ans. $x=7$, $y=9$.

7. Given $\left\{\begin{array}{l}\dfrac{5x+13}{2}-\dfrac{8y-3x-5}{6}=9+\dfrac{7x-3y+1}{3},\\ \dfrac{x+7}{3}:\dfrac{3y-8}{4}+4x::4:21,\end{array}\right\}$ to find the values of x and y.

Ans. $x=5$, $y=4$.

8. Given $\left\{\begin{array}{l}4x-34\frac{1}{3}-\dfrac{4y+13x}{27-6y}=\dfrac{12x+8}{3},\\ 3x+\dfrac{21-4y}{4x-10}=\dfrac{18x+13}{6}-2\frac{1}{9},\end{array}\right\}$ to find the values of x and y.

Ans. $x=7$, $y=5$.

9. Given $\left\{\begin{array}{l}16x+6y-1=\dfrac{128x^2-18y^2+217}{8x-3y+2},\\ \dfrac{10x+10y-35}{2x+2y+3}=5-\dfrac{54}{3x+2y-1},\end{array}\right\}$ to find the values of x and y.

Ans. $x=6$, $y=5$.

10. Given $\left\{\begin{array}{l}4x+3y+\dfrac{24+5\frac{1}{2}y}{2x+1}=\dfrac{16x^2+12xy-8x+5y+28}{4x-2},\\ 2x+4=3y+\dfrac{8x^2-18y^2+108}{4x+6y+3},\end{array}\right\}$ to find the values of x and y.

Ans. $x=3$, $y=2$.

MISCELLANEOUS EXAMPLES IN ELIMINATION

PROBLEM

(281.) 1. Given $\left\{\begin{array}{l}\frac{x}{7}+7y=99 \ (1)\\ \frac{y}{7}+7x=51 \ (2)\end{array}\right\}$ to find the values of x and y.

SOLUTION.

$$\begin{array}{rl} x+49y=693 & (3)=(1)\times 7.\\ 49x+y=357 & (4)=(2)\times 7.\\ x+y=21 & (5)=[(3)+(4)]\div 50.\\ 48y=672 & (6)=(3)-(5).\\ y=\ 14 & (7).\\ x+14=\ 21 & (8)=\text{value of } y \text{ in (7) substituted in (5).}\\ x=7 & \end{array}$$

The above artifice can always be adopted when the coefficients of x and y in the first equation are the same as those of y and x in the second.

PROBLEM

2. Given $\left\{\begin{array}{l}\frac{147}{x}-\frac{147}{x}=28 \ (1)\\ \frac{17}{x}+\frac{56}{y}=\frac{41}{3} \quad (2)\end{array}\right\}$ to find the values of x and y.

SOLUTION.

$$\begin{array}{rl} \frac{1}{x}-\frac{1}{y}=\frac{4}{21} & (3)\ =(1)\div 147.\\ \frac{17}{x}-\frac{17}{y}=\frac{68}{21} & (4)=(3)\times 17.\\ \frac{73}{y}=\frac{73}{7} & (5)=(2)-(4)\\ y=7 & (6).\\ \frac{1}{x}=\frac{4}{21}+\frac{1}{y} & (7)=(3) \text{ trans.}\\ \therefore\ \frac{1}{x}=\frac{4}{21}+\frac{1}{7}=\frac{1}{3}. & \\ x=3. & \end{array}$$

EXAMPLES.

1. Given $\left\{\begin{array}{l}x+\frac{1}{2}y=8\\ \frac{1}{2}x+y=7\end{array}\right\}$ to find the values of x and y.

Ans. $x=6$, $y=4$.

2. Given $\left\{\begin{array}{l}\frac{1}{3}x+3y=21,\\ 3x+\frac{1}{3}y=29,\end{array}\right\}$ to find the values of x and y.

Ans. $x=9$, $y=6$.

3. Given $\left\{\begin{array}{l}\frac{1}{2}x+\frac{1}{3}y=14,\\ \frac{1}{3}x+\frac{1}{2}y=11,\end{array}\right\}$ to find the values of x and y.

Ans. $x=24$, $y=6$.

4. Given $\left\{\begin{array}{l}\frac{1}{8}x+8y=194,\\ 8x+\frac{1}{8}y=131,\end{array}\right\}$ to find the values of x and y.

Ans. $x=16$, $y=24$.

5. Given $\left\{\begin{array}{l}4x+y=34,\\ x+4y=16,\end{array}\right\}$ to find the values of x and y.

Ans. $x=8$, $y=2$.

6. Given $\left\{\begin{array}{l}3x-\frac{1}{2}y=3\frac{1}{2},\\ -x+7y=33,\end{array}\right\}$ to find the values of x and y.

Ans. $x=2$, $y=5$.

7. Given $\left\{\begin{array}{l}x+y=8,\\ x^2-y^2=16,\end{array}\right\}$ to find the values of x and y.

Ans. $x=5$, $y=3$.

8. Given $\left\{\begin{array}{l}x+y=a,\\ x^2-y^2=b,\end{array}\right\}$ to find the values of x and y.

Ans. $x=\dfrac{a^2+b}{2a}, y=\dfrac{a^2-b}{2a}$.

9. Given $\left\{\begin{array}{l}\dfrac{2x}{3}+5y=23,\\ 5x+\dfrac{7y}{4}=-6\frac{1}{4}\end{array}\right\}$ to find the values of x and y.

Ans. $x=-3$, $y=5$.

10. Given $\left\{\begin{array}{l}\dfrac{x}{2}-12=\dfrac{y}{4}+8,\\ \dfrac{x+y}{5}+\dfrac{x}{3}-8=\dfrac{2y-x}{4}+27,\end{array}\right\}$ to find the values of x and y.

Ans. $x=60$, $y=40$.

11. Given $\left\{\begin{array}{l}\frac{1}{2}x+\frac{1}{3}y=8,\\ \frac{1}{3}x-\frac{1}{5}y=-1,\end{array}\right\}$ to find the values of x and y.

Ans. $x=6$, $y=15$.

12. Given $\left\{\begin{array}{l}\dfrac{4x}{x^2}+\dfrac{5y}{y^2}=\dfrac{9}{y}-1,\\ \dfrac{5}{x}+\dfrac{4}{y}=\dfrac{7}{x}+\dfrac{3}{2},\end{array}\right\}$ to find the values of x and y.

Ans. $x=4$, $y=2$.

13. Given $\left\{\begin{array}{l} x-\dfrac{2y-x}{23-x}=20-\dfrac{59-2x}{2}, \\ y+\dfrac{y-3}{x-18}=30-\dfrac{73-3y}{3}, \end{array}\right\}$ to find the values of x and y.

Ans. $x=21$, $y=20$.

14. Given $\left\{\begin{array}{l} \dfrac{3x+2y}{5}-\dfrac{5x-\frac{3}{4}y+1}{3}=x+\dfrac{y-2x}{10}-\dfrac{4x-y}{7}, \\ y+2x : y-2x :: 12x+6y-3 : 6y-12x-1, \end{array}\right\}$ to find the values of x and y.

Ans. $x=1$, $y=4$.

15. Given $\left\{\begin{array}{l} 8x-\dfrac{16+60x}{3y-1}=\dfrac{16xy-107}{5+2y}, \\ 2+6y+9x=\dfrac{27x^2-12y^2+38}{3x-2y+1}, \end{array}\right\}$ to find the values of x and y.

Ans. $x=2$, $y=3$.

16. Given $\left\{\begin{array}{l} 3x+6y+1=\dfrac{6x^2+130-24y^2}{2x-4y+3}, \\ 3x-\dfrac{151-16x}{4y-1}=\dfrac{9xy-110}{3y-4}, \end{array}\right\}$ to find the values of x and y.

Ans. $x=9$, $y=2$.

17. Given $\left\{\begin{array}{l} 3y+11=\dfrac{4x^2-y(x+3y)}{x-y+4}+31-4x, \\ (x+7)(y-2)+3=2xy-(y-1)(x+1), \end{array}\right\}$ to find the values of x and y.

Ans. $x=4$, $y=3$.

18. Given $\left\{\begin{array}{l} \sqrt{y}-\sqrt{y-x}=\sqrt{20-x}, \\ \sqrt{y-x} : \sqrt{20-x} :: 3 : 2, \end{array}\right\}$ to find the values of x and y.

Ans. $x=16$, $y=25$.

19. Given $\left\{\begin{array}{l} 168\frac{15}{44}-19x+\frac{3}{11}y=12\frac{3}{4}x+1084, \\ \frac{1}{3}x-149\frac{1}{2}=319\frac{2}{9}-\frac{7}{2}y, \end{array}\right\}$ to find the values of x and y.

Ans. $x=-27\frac{2}{3}$, $y=136\frac{5}{9}$.

20. Given $\left\{\begin{array}{l} 3x+5y=\dfrac{(8a-2b)ab}{a^2-b^2}, \\ a^2x-\dfrac{ab^2c}{a+b}+(a+b+c)by=b^2x+(a+2b)ab, \end{array}\right\}$ to find the values of x and y.

Ans. $x=\dfrac{ab}{a-b}$, $y=\dfrac{ab}{a+b}$.

21. Given $\left\{\begin{array}{l} bcx=cy-2b, \\ b^2y+\dfrac{a(c^3-b^3)}{bc}=\dfrac{2b^3}{c}+c^3x, \end{array}\right\}$ to find the values of x and y.

Ans. $\frac{a}{bc}$, $y=\frac{a+2b}{c}$.

22. Given $\left\{\begin{array}{l} \dfrac{a}{x}+\dfrac{b}{y}=m, \\ \dfrac{c}{x}+\dfrac{d}{y}=n, \end{array}\right\}$ to find the values of x and y.

Ans. $x=\frac{bc-ad}{nb-md}$, $y=\frac{bc-ad}{mc-na}$.

23. Given $\left\{\begin{array}{l} \dfrac{6x+9}{4}+\dfrac{3x+5y}{4x-6}=3\frac{1}{4}+\dfrac{3x+4}{2}, \\ \dfrac{8y+7}{10}+\dfrac{6x-3y}{2y-8}=4+\dfrac{4y-9}{5}, \end{array}\right\}$ to find the values of x and y.

Ans. $x=7$, $y=9$.

24. Given $\left\{\begin{array}{l} \frac{2}{3}(x-\frac{3}{5}y)+\dfrac{x+\frac{1}{5}y}{6}=\frac{1}{3}-\frac{1}{2}\left\{\dfrac{\frac{4}{5}y-2}{6}-(x-y)\right\}, \\ x-2y-\dfrac{3y-5x}{2}=\frac{11}{2}(x+y)-3(x-y), \end{array}\right\}$

to find the values of x and y. *Ans.* $x=\frac{10}{7}$, $y=\frac{5}{42}$.

25. Given $\left\{\begin{array}{l} 4+\dfrac{12y-\dfrac{6y+2}{x}}{11}=y+\dfrac{\dfrac{3xy-31}{11}+10x+13}{3x}, \\ \dfrac{2x}{3}-\dfrac{3x-5}{y+7}=\dfrac{4xy+\frac{170}{3}}{6y+27}, \end{array}\right\}$ to find

the values of x and y. *Ans.* $x=7$, $y=2$.

26. Given $\left\{\begin{array}{l} \dfrac{\dfrac{7x}{4}+6y}{5}-\dfrac{\dfrac{3y+6}{5}-\dfrac{3x-2}{10}}{8}=5-\dfrac{x}{16}, \\ \dfrac{3x}{2}+\dfrac{2y}{3}+2\frac{1}{2} : \dfrac{x}{2}-\dfrac{y}{3}+\frac{1}{6} :: 10\frac{1}{2} : 1\frac{1}{6}, \end{array}\right\}$ to find the values of x and y.

Ans. $x=4$, $y=3$.

27. Given $\left\{\begin{array}{l} \dfrac{4x-2y+3}{3}-\dfrac{18-x+5y}{7}=\dfrac{x}{4}-\dfrac{y}{5}-\dfrac{1}{7}-7\frac{7}{10}, \\ 2x-y+15 : y-2x+15 :: \dfrac{x}{3}-\dfrac{y}{4}+\dfrac{3}{4} : \dfrac{y}{4}-\dfrac{x}{3}+\dfrac{1}{12}, \end{array}\right\}$

to find the values of x and y. *Ans.* $x=18$, $y=24$.

28. Given $\left\{\begin{array}{l} \dfrac{x-6}{7y}+\dfrac{4x+7}{24}-\dfrac{x-\frac{y}{7}}{6}=\dfrac{19+y}{42}-\dfrac{\frac{11x}{3}+6}{56y}, \\ 12x-15y+\frac{13}{4} : 10y-8x+\frac{86}{3} :: 93-9x : 6x-\frac{14}{5}, \end{array}\right\}$

to find the values of x and y. *Ans.* $x=9,\ y=7$.

29. Given $\left\{\begin{array}{l} \dfrac{3x-5y}{3}-\dfrac{2x-8y-9}{12}=\dfrac{y}{12}+\frac{1}{3}+\frac{1}{4}, \\ \dfrac{x}{7}+\dfrac{y}{4}+1\frac{1}{3} : 4x-\dfrac{y}{8}-24 :: 3\frac{1}{3} : 3\frac{1}{2}, \end{array}\right\}$ to find the values of x and y.

Ans. $x=7,\ y=4$.

30. Given $\left\{\begin{array}{l} x-\dfrac{3y-2+x}{11}=1+\dfrac{15x+\frac{4y}{3}}{33}, \\ \dfrac{3x+2y}{6}-\dfrac{y-5}{4}=\dfrac{11x+152}{12}-\dfrac{3y+1}{2}, \end{array}\right\}$ to find the values of x and y.

Ans. $\left\{\begin{array}{l} x=8. \\ y=9. \end{array}\right.$

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE CONTAINING THREE OR MORE UNKNOWN QUANTITIES.

PROBLEM.

(282.) To eliminate from three or more simultaneous equations.

SOLUTION.

Let (A), (B), and (C), represent three simple equations, each of which contains three unknown quantities, as x, y, and z. Suppose it most convenient first to eliminate z. Then, according to one of the preceding methods, eliminate z from (A) and (B), and there will result an equation which will contain only x and y as unknown quantities, which equation designate by (D). Next, eliminate z from (A) and (C), or from (B) and (C), as may be most convenient, and there will result another equation, which also will contain only x and y as unknown quantities. Call this equation (E). We have now nothing

more to do with (A), (B), and (C), but must operate on the resulting equations, (D) and (E). Now, let us eliminate y from (D) and (E), and there will result an equation (F) which will contain only x. *Q. E. F.*

PROBLEM

(283.) 1. Given $\left\{\begin{array}{l} 2x+3y+5z=23 \ (1), \\ 3x+2y+7z=28 \ (2), \\ 5x+7y+11z=52 \ (3), \end{array}\right\}$ to find the values of x, y, and z.

SOLUTION.

$$\begin{array}{ll}
14x+21y+35z=161 & (4)=(1)\times 7. \\
15x+10y+35z=140 & (5)=(2)\times 5. \\ \hline
x-11y=-21 & (6)=(5)-(4). \\
22x+33y+55z=253 & (7)=(1)\times 11. \\
25x+35y+55z=260 & (8)=(3)\times 5. \\ \hline
3x+2y=7 & (9)=(8)-(7). \\
3x-33y=-63 & (10)=(6)\times 3. \\ \hline
35y=70 & (11)=(9)-(10). \\
y=2 & (12)=(11)\div 35. \\
3x=7-2y & (13)=(9) \text{ transposed.} \\
\therefore\ 3x=7-4=3 & (14)=\text{value of } y \text{ substituted in } (13). \\
x=1 & (15)=(14)\div 3. \\
2+6+5z=23 & (16)=\text{values of } x \text{ and } y \text{ substituted in } (1). \\
5z=15 & \\
z=3. &
\end{array}$$

PROBLEM

2. Given $\left\{\begin{array}{l} ax+by+cz=d, \\ a'x+b'y+c'z=d', \\ a''x+b''y+c''z=d'', \end{array}\right\}$ to find the values of x, y, and z.

SOLUTION.

Eliminating as directed in Problem **(282)**, we shall obtain, after arranging the terms in the separate results,

$$x=\frac{db'c''+d'b''c+d''bc'-d'bc''-db''c'-d''b'c}{ab'c''+a'b''c+a''bc'-a'bc''-ab''c'-a''b'c}$$

$$y=\frac{ad'c''+a'd''c+a''dc'-a'dc''-ad''c'-a''d'c}{ab'c''+a'b''c+a''bc'-a'bc''-ab''c'-a''b'c}$$

$$z=\frac{ab'd''+a'b''d+a''bd'-a'bd''-ab''d'-a''b'd}{ab'c''+a'b''c+a''bc'-a'bc''-ab''c'-a''b'c}$$

If the student can only recollect these general values, he need only make the proper substitutions without being compelled to go through the process of elimination.

The denominators are all alike, and do not contain any of the absolute terms. This denominator can be formed as follows:

Write the coefficients of the unknown quantities as in the problem, repeating the first two rows; thus,

$$\begin{matrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \\ a & b & c \\ a' & b' & c' \end{matrix}$$

Omitting $\begin{matrix} b & c \\ & c' \end{matrix}$, at the right hand corner above, and $\begin{matrix} a & \\ a' & b' \end{matrix}$, at the left hand corner below, we have,

$$\begin{array}{llll} \text{1st.} & a & & \\ \text{2d.} & a' & b' & \\ \text{3d.} & a'' & b'' & c'' \\ & & b & c \\ & & & c' \end{array}$$

The product of the terms in each of these lines will give the positive terms of the common denominator.

Again, omitting $\begin{matrix} a & b \\ a' & \end{matrix}$, at the left hand corner above, and $\begin{matrix} & c \\ b' & c' \end{matrix}$, at the right hand corner below, we have

$$\begin{array}{llll} & & & c \\ & & b' & c' \\ \text{6th.} & a'' & b'' & c \\ \text{5th.} & a & b & \\ \text{4th.} & a' & & \end{array}$$

The product of the terms in each of these lines will give the negative terms of the common denominator.

We can write the respective numerators from the common denominator by changing a into d, a' into d', and a'' into d'' for the numerator in the value of x; b into d, b' into d', and b'' into d'' for the numerator in the value of y; and c into d, c' into d', and c'' into d'' for the numerator in the value of z.

It is not necessary that the student should in practice make the omissions which we did above, for we can get from

$$\begin{array}{ccc} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \\ a & b & c \\ a' & b' & c' \end{array}$$

by passing downward to the right, the terms $ab'c''$, $a'b''c$, $a''bc'$, and by passing upward to the right, the terms $a'bc''$, $ab''c'$, $a''b'c$.

In the same manner, by putting d for a, d' for a', and d'' for a'' we should have from

$$\begin{array}{ccc} d & b & c \\ d' & b' & c' \\ d'' & b'' & c'' \\ d & b & c \\ d' & b' & c' \end{array}$$

by passing downward to the right, $db'c''$, $d'b''c$, and $d''bc'$ for the positive terms in the numerator of the value of x, and by passing upward to the right, $d'bc''$, $db''c'$, and $d''b'c$ for the negative terms in the same numerator.

In the same way from

$$\begin{array}{ccc} a & d & c \\ a' & d' & c' \\ a'' & d'' & c'' \\ a & d & c \\ a' & d' & c' \end{array}$$

we can get the numerators in the value of y.

Also, from

$$\begin{array}{ccc} a & b & d \\ a' & b' & d' \\ a'' & b'' & d'' \\ a & b & d \\ a' & b' & d' \end{array}$$

we can get the numerator in the value of z

PROBLEM

3. Given $\left\{\begin{array}{l} 3x+2y+5z=59, \\ 4x+\ \ y+3z=41, \\ 8x+7y+2z=75, \end{array}\right\}$ to find the values of x, y, and z.

SOLUTION.

From

$$\begin{array}{ccc} 3 & 2 & 5 \\ 4 & 1 & 3 \\ 8 & 7 & 2 \\ 3 & 2 & 5 \\ 4 & 1 & 3 \end{array}$$

we get $3\times1\times2+4\times7\times5+8\times2\times3-4\times2\times2-3\times7\times3-8\times1\times5$ for the common denominator of x, y, and z.

From
$$\begin{array}{ccc} 59 & 2 & 5 \\ 41 & 1 & 3 \\ 75 & 7 & 2 \\ 59 & 2 & 5 \\ 41 & 1 & 3 \end{array}$$
we get for the numerator in the value of x,
$$59\times1\times2+41\times7\times5+75\times2\times3-41\times2\times2-59\times7\times3-75\times1\times5.$$

These expressions simplified, give $x=\dfrac{2003-1778}{194-119}=\dfrac{225}{75}=3.$

From
$$\begin{array}{ccc} 3 & 59 & 5 \\ 4 & 41 & 3 \\ 8 & 75 & 2 \\ 3 & 59 & 5 \\ 4 & 41 & 3 \end{array}$$

We get for the numerator in the value of y,
$$3\times41\times2+4\times75\times5+8\times59\times3-4\times59\times2-3\times75\times3-8\times41\times5=3162-2787.$$

Therefore, $y=\dfrac{3162-2787}{194-119}=\dfrac{375}{75}=5.$

Also, from
$$\begin{array}{ccc} 3 & 2 & 59 \\ 4 & 1 & 41 \\ 8 & 7 & 75 \\ 3 & 2 & 59 \\ 4 & 1 & 41 \end{array}$$

We get for the numerator in the value of z
$$3\times1\times75+4\times7\times59+8\times2\times41-4\times2\times75-3\times7\times41-8\times1\times59=2533-1933.$$

Therefore, $z=\dfrac{2533-1933}{194-119}=\dfrac{600}{75}=8.$

PROBLEM

4. Given $\left\{\begin{array}{l} 3x-6y+z=-32, \\ 2x+8y \quad = \ 16, \\ 5x+7y-2z= \ 5, \end{array}\right\}$ to find the value of x, y and z.

SOLUTION.

From $\left.\begin{array}{rrr} 3 & -6 & 1 \\ 2 & 8 & 0 \\ 5 & 7 & -2 \\ 3 & -6 & 1 \\ 2 & 8 & 0 \end{array}\right\}$, $\left.\begin{array}{rrr} -32 & -6 & 1 \\ 16 & 8 & 0 \\ 5 & 7 & -2 \\ -32 & -6 & 1 \\ 16 & 8 & 0 \end{array}\right\}$, $\left.\begin{array}{rrr} 3 & -32 & 1 \\ 2 & 16 & 0 \\ 5 & 5 & -2 \\ 3 & -32 & 1 \\ 2 & 16 & 0 \end{array}\right\}$, and $\begin{array}{rrr} 3 & -6 & -32 \\ 2 & 8 & 16 \\ 5 & 7 & 5 \\ 3 & -6 & -32 \\ 2 & 8 & 16 \end{array}$

We get by proceeding as before

$$x=\frac{(-32\times8\times-2+16\times7\times1+5\times-6\times0)-(16\times-6\times-2+-32\times7\times0+5\times8\times1)}{(3\times8\times-2+2\times7\times1+5\times-6\times0)-(2\times-6\times-2+3\times7\times0+5\times8\times1)}$$

$$=\frac{624-232}{-34-64}=\frac{392}{-98}=-4.$$

$$y=\frac{(3\times16\times-2+2\times5\times1+5\times-32\times0)-(2\times-32\times-2+3\times5\times0+5\times16\times1)}{D}$$

$$=\frac{-86-208}{D}=\frac{-294}{-98}=3.$$

$$z=\frac{(3\times8\times5+2\times7\times-32+5\times-6\times16)-(2\times-6\times5+3\times7\times16+5\times8\times-32)}{D}$$

$$=\frac{-808+1004}{D}=\frac{196}{-98}=-2.$$

PROBLEM

5. Given $\left\{\begin{array}{l} x+y+z=31, \ (1) \\ x+y-z=25, \ (2) \\ x-y-z=\ 9, \ (3) \end{array}\right\}$ to find the values of x, y and z.

SOLUTION.

$2x=40$	$(4)=(1)+(3)$.
$x=20$	(5).
$2z=\ 6$	$(6)=(1)-(2)$.
$z=\ 3$	(7).
$2y=16$	$(8)=(2)-(3)$.
$y=\ 8$.	

PROBLEM

6. Given $\left\{\begin{array}{r} 2x-3y+2z=13, \ (1) \\ 4v-2x=30, \ (2) \\ 4y+2z=14, \ (3) \\ 5y+3v=32, \ (4) \end{array}\right\}$ to find the values of v, x, y, and z.

SOLUTION.

$7y-2x=\ 1$	$(5)=(3)-(1)$.
$12v-6x=\ 90$	$(6)=(2)\times3$.
$20y+12v=128$	$(7)=(4)\times4$.
$20y+6x=\ 38$	$(8)=(7)-(6)$.
$21y-6x=\ \ 3$	$(9)=(5)\times3$.
$41y=\ 41$	$(10)=(8)+(9)$.
$y=\ \ 1$	(11).
$2x=7y-1$	$(12)=(5)$ transposed.
$2x=\ 7-1=6$	(13).
$x=\ 3$	(14).

[forward

$$4v = 2x + 30 \qquad (15) = (2) \text{ transposed.}$$
$$4v = 6 + 30 = 36 \qquad (16).$$
$$v = 9 \qquad (17) = (16) \div 4.$$
$$2z = 14 - 4y \qquad (18) = (3) \text{ transposed.}$$
$$2z = 14 - 4 = 10.$$
$$z = 5.$$

NOTE.—The student should accustom himself to apply the general formula which has been given, so that it may be called up at any time. It will often save much labor, as will be proved to be the case in some of the following examples. There are many curious properties which belong to the general formula for the values of the unknown quantities as deduced from three simultaneous equations. But we leave them to be discovered by the student.

EXAMPLES.

1. Given $\left\{\begin{array}{l} x+y+z=29, \\ x+2y+3z=62, \\ \frac{x}{2}+\frac{y}{3}+\frac{z}{4}=10, \end{array}\right\}$ to find the values of x, y, and z.

Ans. $x=8$, $y=9$, $z=12$.

2. Given $\left\{\begin{array}{l} x+2y+3z=14, \\ 2x-3y+4z=8, \\ 3x+4y-5z=-4, \end{array}\right\}$ to find the values of x, y, and z.

Ans. $x=1$, $y=2$, $z=3$.

3. Given $\left\{\begin{array}{l} bz+cy=a. \\ az+cx=b, \\ ay+bx=c, \end{array}\right\}$ to find the values of x, y, and z.

Ans. $\left\{\begin{array}{l} x=\dfrac{b^2+c^2-a^2}{2bc}. \\ y=\dfrac{a^2+c^2-b^2}{2ac}. \\ z=\dfrac{a^2+b^2-c^2}{2ab}. \end{array}\right.$

4. Given $\left\{\begin{array}{l} ax+by=a^2, \\ bx-az=b^2, \\ cx+dv=c^2, \\ dy-cz=d^2, \end{array}\right\}$ to find the values of v, x, y, and z.

Ans. $\left\{\begin{array}{l} x=\dfrac{a^3d+b^3c-abd^2}{a^2d+b^2c}. \\ y=\dfrac{a^2d^2+a^2bc-ab^2c}{a^2d+b^2c}. \\ z=\dfrac{(a^2-ab-bd)bd}{a^2d+b^2c}. \\ v=\dfrac{(ac+bd-a^2)acd-(b-c)b^2c^2}{(a^2d+b^2c)d}. \end{array}\right.$

5. Given $\left\{\begin{array}{l} x+y=a, \\ x+z=b, \\ y+z=c, \end{array}\right\}$ to find the values of x, y, and z.

Ans. $\left\{\begin{array}{l} x=\frac{1}{2}(a+b-c). \\ y=\frac{1}{2}(a-b+c). \\ z=\frac{1}{2}(c-a+b). \end{array}\right.$

6. Given $\left\{\begin{array}{l} x-y-z=6, \\ 3y-x-z=12, \\ 7z-y-x=24, \end{array}\right\}$ to find the values of x, y, and z.

Ans. $x=39$, $y=21$, $z=12$.

7. Given $\left\{\begin{array}{l} x+y-z=8, \\ 2x-y+3z=21, \\ 4z+3y-2x=17, \end{array}\right\}$ to find the values of x, y, and z.

Ans. $x=7$, $y=5$, $z=4$.

8. Given $\left\{\begin{array}{l} \frac{1}{x}+\frac{1}{y}=\frac{5}{6}, \\ \frac{1}{x}+\frac{1}{z}=\frac{3}{4}, \\ \frac{1}{y}+\frac{1}{z}=\frac{7}{12}, \end{array}\right\}$ to find the values of x, y, and z.

Ans. $x=2$, $y=3$, $z=4$.

9. Given $\left\{\begin{array}{l} x+2y+3z=17, \\ y+2z+3x=13, \\ z+2x+3y=12, \end{array}\right\}$ to find the values of x, y, and z.

Ans. $x=1$, $y=2$, $z=4$.

10. Given $\left\{\begin{array}{l} x+a(y+z)=m, \\ y+b(x+z)=n, \\ z+c(x+y)=p, \end{array}\right\}$ to find the values of x, y, and z.

Ans. $\left\{\begin{array}{l} x=\dfrac{m+nca+pab-na-mcb-pa}{1+2abc-ba-cb-ca}. \\ y=\dfrac{n+pab+mbc-pb-nac-mb}{1+2ab-ba-cb-ca}. \\ z=\dfrac{p+mbc+nca-mc-pba-nc}{1+2abc-ba-cb-ca}. \end{array}\right.$

11. Given $\left\{\begin{array}{l} x-y+z=30, \\ 8x-4y+2z=50, \\ 27x-9y+3z=64, \end{array}\right\}$ to find the values of x, y, and z.

Ans. $x=\frac{2}{3}$, $y=7$, $z=36\frac{1}{3}$.

12. Given $\left\{\begin{array}{l} \frac{4x+3y+z}{10}-\frac{2y+2z-x+1}{15}=5+\frac{x-z-5}{5}, \\ \frac{9x+5y-2z}{12}-\frac{2x+y-3z}{4}=\frac{7y+z+3}{11}+\frac{1}{6}, \\ \frac{5y+3z}{4}-\frac{2x+3y-z}{12}+2z=y-1+\frac{3x+2y+7}{6}, \end{array}\right\}$ to find the values of x, y, and z. *Ans.* $x=9$, $y=7$, $z=3$.

13. Given $\left\{\begin{array}{l} \frac{x}{6}-\frac{y}{5}+\frac{z}{4}=11, \\ \frac{x}{5}+\frac{y}{4}-\frac{z}{3}=35, \\ \frac{x}{4}-\frac{y}{3}+\frac{z}{2}=16, \end{array}\right\}$ to find the values of x, y, and z.

Ans. $x=120$, $y=60$, $z=12$.

14. Given $\left\{\begin{array}{l} 11x-10y=\frac{12y-11z}{3}, \\ \frac{x+z-2y}{3}=\frac{z-y-1}{2}, \\ 3x=y+z+7, \end{array}\right\}$ to find the values of x, y, and z.

Ans. $x=10$, $y=11$, $z=12$.

15. Given $\left\{\begin{array}{l} 3x-y+z=15, \\ 5x+3y-2z=16, \\ 7x+4y-5z=11, \end{array}\right\}$ to find the values of x, y, and z.

Ans. $x=4$, $y=2$, $z=5$.

16. Given $\left\{\begin{array}{l} 2x+4y-3z=22, \\ 4x-2y+5z=18, \\ 6x+7y-z=63, \end{array}\right\}$ to find the values of x, y, and z.

Ans. $x=3$, $y=7$, $z=4$.

17. Given $\left\{\begin{array}{l} 3x+2y-z=20, \\ 2x+3y+6z=70, \\ x-y+6z=41, \end{array}\right\}$ to find the values of x, y, and z.

Ans. $x=5$, $y=6$, $z=7$.

18. Given $\left\{\begin{array}{l} 7x+12y+4z=128, \\ 3x+3y+7z=60, \\ 6x+y+5z=68, \end{array}\right\}$ to find the values of x, y, and z.

Ans. $x=8$, $y=5$, $z=3$.

19. Given $\left\{\begin{array}{l} 6x+3y-4z=22, \\ 4x-y+6z=20, \\ 5x+2y-6z=11, \end{array}\right\}$ to find the values of x, y, and z.

Ans. $x=3$, $y=4$, $z=2$.

20. Given $\left\{\begin{array}{l} 2x=u+y+z, \\ 3y=u+x+z, \\ 4z=u+x+y, \\ x=u+14, \end{array}\right\}$ to find the values of u, x, y, and z.

Ans. $u=26$, $x=40$, $y=30$, $z=24$.

21. Given $\left\{\begin{array}{r} x-y-z=-a, \\ -2x+y-2z=-a, \\ -3x-3y+z=-a, \end{array}\right\}$ to find the values of x, y, and z.

Ans. $x=\frac{1}{11}a$, $y=\frac{5}{11}a$, $z=\frac{7}{11}a$.

22. Given $\left\{\begin{array}{r} 2x+y-2z=40, \\ 4y-x+3z=35, \\ 3u+t=13, \\ y+u+t=15, \\ 3x-y+3t-u=49, \end{array}\right\}$ to find the values of x, y, z, u, and t.

Ans. $x=20$, $y=10$, $z=5$, $u=4$, $t=1$.

23. Given $\left\{\begin{array}{l} u+v+x+y=10, \\ u+v+x+z=11, \\ u+v+y+z=12, \\ u+x+y+z=13, \\ v+x+y+z=14, \end{array}\right\}$ to find the values of u, v, x, y, and z.

Ans. $u=1$, $v=2$, $x=3$, $y=4$, $z=5$.

24. Given $\left\{\begin{array}{l} \dfrac{x+y}{3}+2z=21, \\ \dfrac{y+z}{2}-3x=-65, \\ \dfrac{3x+y-z}{2}=38, \end{array}\right\}$ to find the values of x, y, and z.

Ans. $x=24$, $y=9$, $z=5$.

25. Given $\left\{\begin{array}{r} x+\frac{1}{2}y+\frac{1}{3}z=32, \\ \frac{1}{3}x+\frac{1}{4}y+\frac{1}{5}z=15, \\ \frac{1}{4}x+\frac{1}{5}y+\frac{1}{6}z=12, \end{array}\right\}$ to find the values of x, y, and z.

Ans. $x=12$, $y=20$, $z=30$.

26. Given $\left\{\begin{array}{r} 3x-9y+8z=41, \\ -5x+4y+2z=-20, \\ 11x-7y-6z=37, \end{array}\right\}$ to find the values of x, y, and z.

Ans. $x=2$, $y=-3$, $z=1$.

27. Given $\left\{\begin{array}{r} 7x+5y+2z=79, \\ 8x+7y+9z=122, \\ x+4y+5z=55, \end{array}\right\}$ to find the values of x, y, and z.

Ans. $x=4$, $y=9$, $z=3$.

28. Given $\left\{\begin{aligned} x+y+z&=a \\ my&=nx \\ pz&=qx \end{aligned}\right\}$ to find the values of x, y, and z.

$$\textit{Ans.}\ x=\frac{amp}{mp+np+mq},\ y=\frac{anp}{mp+np+mq},\ z=\frac{amq}{mp+np+mq}.$$

29. Given $\left\{\begin{aligned} \frac{1}{x}+\frac{1}{y}&=\frac{1}{a}, \\ \frac{1}{x}+\frac{1}{z}&=\frac{1}{b}, \\ \frac{1}{y}+\frac{1}{z}&=\frac{1}{c}, \end{aligned}\right\}$ to find the values of x, y, and z.

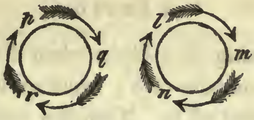
$$\textit{Ans.}\ x=\frac{2abc}{ac-ab+bc},\ y=\frac{2abc}{ab-ac+bc},\ z=\frac{2abc}{ac+ab-bc}.$$

30. Given $\left\{\begin{aligned} x+ly+lz&=p, \\ mx+y+mz&=q, \\ nx+ny+z&=r, \end{aligned}\right\}$ to find the values of x, y, and z.

$$\textit{Ans.}\ \left\{\begin{aligned}
x&=\frac{p}{1-l}-\frac{l}{1-l}\left\{\frac{\dfrac{p}{1-l}+\dfrac{q}{1-m}+\dfrac{r}{1-n}}{1+\dfrac{l}{1-l}+\dfrac{m}{1-m}+\dfrac{n}{1-n}}\right\}, \\
y&=\frac{q}{1-m}-\frac{m}{1-m}\left\{\frac{\dfrac{p}{1-l}+\dfrac{q}{1-m}+\dfrac{r}{1-n}}{1+\dfrac{l}{1-l}+\dfrac{m}{1-m}+\dfrac{n}{1-n}}\right\}, \\
z&=\frac{r}{1-n}-\frac{n}{1-n}\left\{\frac{\dfrac{p}{1-l}+\dfrac{q}{1-m}+\dfrac{r}{1-n}}{1+\dfrac{l}{1-l}+\dfrac{m}{1-m}+\dfrac{n}{1-n}}\right\}.
\end{aligned}\right.$$

NOTE.—This example is the same as Ex. 10, but the answer is in another form, indicating that it has been solved in a different manner. The student may observe that the expressions in the brackets are identical. Also, in the value of x, the letters which are not included in the brackets are only those which occur in the first equation; in the value of y, only those that occur in the second equation; and in the value of z, only those that occur in the third equation.

Because these values are *symmetrical*, being derived from *symmetrical equations*, we can, after getting the value of x, deduce the value of y from it, and, having the value of y, we can, in like manner, deduce the value of z. If in the value of x, we change each letter to that which is in advance of it in the circles,



we shall get the value of y. In like manner, changing the letters in the value of y, we should get the value of z. The quantities in the brackets would still be the same, though thus permuted, only the terms would not occur in the same order.

QUESTIONS INVOLVING SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE.

QUESTION

(284.) 1. A man and his wife could drink a barrel of beer in 15 days. After drinking together 6 days, the woman alone drank the remainder in 30 days. In what time would either, alone, drink a barrel?

SOLUTION.

Let x = the number of days in which the man could drink it,
And y = " " " woman " "

$$\text{Then } \frac{1}{x}+\frac{1}{y}=\frac{1}{15}.$$

In 6 days both drank $\frac{6}{15}=\frac{2}{5}$, leaving $\frac{3}{5}$. It took the woman 30 days to drink this $\frac{3}{5}$; therefore, in 1 day she would drink $\frac{1}{30}$ of $\frac{3}{5}=\frac{1}{50}$.

$$\text{Hence, } \frac{1}{y}=\frac{1}{50}$$

$$y=50$$

$$\frac{1}{x}+\frac{1}{50}=\frac{1}{15}$$

$$\frac{1}{x}=\frac{7}{150}$$

$$x=\frac{150}{7}=21\tfrac{3}{7}.$$

QUESTION

2. A number consisting of 2 digits when divided by 4, gives a certain quotient and a remainder of 3; when divided by 9, gives another quotient and a remainder of 8. Now, the *value* of the digit on the left hand is equal to the quotient which was obtained when the number was divided by 9; and the other digit is equal to $\frac{1}{17}$ of the quotient obtained when the number was divided by 4. What is the number?

SOLUTION.

Let $x =$ the digit in the tens' place,
And $y =$ " " units' "

Then, in consequence of our system of notation, $10x+y$ must represent the number. Since $10x+y$ divided by 4 leaves a remainder of 3, if we subtract 3 from $10x+y$ the result is exactly divisible by 4, and by the question the quotient is $17y$.

Hence, we have the equation,

$$\frac{10x+y-3}{4}=17y.$$ In like manner, we get

$$\frac{10x+y-8}{9}=x$$

$$\begin{aligned} 10x-67y &= 3 \\ 10x+10y &= 80 \\ 77y &= 77 \\ y &= 1 \\ x+1 &= 8 \\ x &= 7 \end{aligned}$$

Therefore, 71 is the number, because it is equal to $10x+y$.

QUESTION

3. What two numbers are there in the ratio of 5 to 7, to which, if two other required numbers in the ratio of 3 to 5 be added, the sums shall be in the ratio of 9 to 13, and the difference of these sums shall be 16?

SOLUTION.

Let $5x$ and $7x$ be the numbers in the ratio of 5 to 7,
And $3y$ and $5y$ " " " 3 to 5.

$$\text{Then, } 5x+3y : 7x+5y :: 9 : 13$$
$$\text{Or, } 65x+39y=63x+45y$$
$$2x=6y$$
$$x=3y$$

But $(7x+5y)-(5x+3y)$, or $2x+2y=16$
$$x+y=8$$
Whence, because $x=3y$, $\quad 3y+y=8$
$$4y=8$$
$$y=2.$$
But, since $x=3y$ and $y=2$, we have $\quad x=6.$

Hence, $5x$ and $7x$, or the first two numbers are 30 and 42; and $3y$ and $5y$, or the other two numbers are 6 and 10.

QUESTIONS.

1. What two numbers are there, the greater of which is to the less as their sum is to 42, and their difference is to 6? *Ans.* 32 and 24.

2. A person expends half a crown, or 30 pence, in apples and pears, buying his apples at 4, and his pears at 5 for a penny; and afterward accommodates his neighbor with half his apples, and one-third of his pears for 13 pence. How many did he buy of each?
Ans. 72 apples, and 60 pears.

3. A farmer sells to one person 9 horses and 7 cows for \$600; and to another, at the same prices, 6 horses and 13 cows for the same sum. What was the price of each?
Ans. The price of a cow was \$24, and of a horse \$48.

4. A farmer hires a farm for \$245 per annum; the arable land being valued at \$2 an acre, and the pasture at \$1.40; now the number of acres of arable is to half the excess of the arable above the pasture as 28 : 9. How many acres were there of each?
Ans. 98 acres of arable, and 35 of pasture.

5. There is a number consisting of two digits, the second of which is greater than the first; and if the number be divided by the sum of its digits, the quotient is 4; but if the digits be inverted, and that number divided by a number greater by 2 than the difference of the digits, the quotient becomes 14. What is the number? *Ans.* 48.

6. What fraction is that, whose numerator being doubled, and denominator increased by 7, the value becomes $\frac{2}{3}$; but the denominator being doubled, and the numerator increased by 2, the value becomes $\frac{3}{5}$? *Ans.* $\frac{4}{5}$.

7. Two persons, A and B can perform a piece of work in 16 days. They work together 4 days, when A being called off, B is left to finish it, which he does in 36 days more. In what time could each do it separately? *Ans.* A in 24 days, and B in 48 days.

8. There is a cistern, into which water is admitted by three pipes, two of which are exactly of the same dimensions. When they are all open, $\frac{5}{12}$ of the cistern is filled in 4 hours; and if one of the equal pipes be stopped, $\frac{7}{9}$ of the cistern is filled in 10 hours and 40 minutes. In how many hours would each pipe fill the cistern?
Ans. Each of the equal ones in 32 hours, and the other in 24.

9. Some hours after a courier had been sent from A to B, which are 147 miles distant, a second was sent, who wished to overtake him just as he entered B; in order to do this he must perform the journey in 28 hours less than the first did. Now the time in which the first travels 17 miles added to the time in which the second travels 56 miles is 13 hours and 40 minutes. How many miles does each go per hour? *Ans.* The 1st 3, and the 2d 7 miles an hour.

10. A and B playing at backgammon; A bet 3 dimes to 2 dimes on every game, and after a certain number of games found that he had lost 17 dimes. Now had A won 3 more from B, the number he would then have won would have been to the number B would have won as 5 to 4. How many games did they play? *Ans.* 9.

11. A and B engaged to reap a field of corn in 12 days. The times in which they could severally reap an acre are as 2 : 3. After some time, finding themselves unable to finish it in the stipulated time, they called in C to help them; whose rate of working was such, that if he had worked with them from the beginning, it would have been finished in 9 days. Also, the times in which he could severally have reaped the field with A alone, and with B alone, are in the ratio of 7 to 8. When was C called in? *Ans.* After 6 days.

12. A vintner has 2 casks of wine, from the greater of which he draws 15 gallons, and from the less 11; and finds the quantities remaining to be in the ratio of 8 to 3. After they become half empty, he puts 10 gallons of water into each, and finds that the quantities of liquid now in them are as 9 to 5. How many gallons will each hold? *Ans.* The larger 79, and the smaller 35 gallons.

13. At an election for two members of parliament, three men offer themselves as candidates, and all the electors give single votes. The

number of voters for the two successful ones are in the ratio of 9 to 8; and if the first had had seven more, his majority over the second would have been to the majority of the second over the third as 12 : 7. Now if the first and third had formed a coalition, and had one more voter, they would each have succeeded by a majority of 7. How many voted for each? *Ans.* 369, 328, and 300, respectively.

14. A wine merchant has two kinds of wine. If he mixes a gallons of the first with b gallons of the second, the mixture is worth c dollars per gallon; but if he mixes f gallons of the first with g gallons of the second, the mixture is worth h dollars per gallon. What is the price of each kind of wine per gallon?

$$\textit{Ans.}\left\{\begin{array}{l}\text{The price of the first kind is } \dfrac{(a+b)cg-(f+g)bh}{ag-bf} \\ \text{“ \quad “ \quad “ \quad second kind } \dfrac{(a+b)cf-(f+g)ah}{bf-ag}\end{array}\right\}\begin{array}{c}\text{dollars}\\ \text{per}\\ \text{gallon.}\end{array}$$

15. A banker has two kinds of money; it takes a pieces of the first to make a crown, and b pieces of the second to make the same amount. Some one gave him a crown for c pieces. How many pieces of each kind did the person receive?

Ans. $\dfrac{a(c-b)}{a-b}$ pieces of the 1st kind, and $\dfrac{b(a-c)}{a-b}$ of the 2d kind.

16. What two fractions added make $\frac{3}{5}$, and the sum of whose numerators is equal to the sum of their denominator?

Ans. $\frac{1}{2}$ and $\frac{1}{10}$.

17. A purse holds 19 crowns and 6 guineas. Now 4 crowns and 5 guineas fill $\frac{17}{63}$ of it. How many will it hold of each?

Ans. 21 crowns, or 63 guineas.

18. \$500 was to be lent out at simple interest in two separate sums, the smaller, at 2 per cent. more than the other. The interest of the greater sum was afterward increased, and that of the smaller diminished by 1 per cent. By this, the interest of the whole was augmented by one-fourth of the former value. But if the interest of the greater sum had been so increased, without any diminution of the less, the interest of the whole would have been increased one-third. What were the sums and the rate per cent. of each?

Ans. \$100 and \$400, and 4 and 2 per cent. respectively.

19. Some smugglers discovered a cave, which would exactly hold the cargo of their boat; viz. 13 bales of cotton, and 33 casks of rum.

Whilst they were unloading, a custom-house cutter coming in sight, they sailed away with 9 casks and 5 bales, leaving the cave two-thirds full. How many bales, or casks would it hold?

Ans. 24 bales, or 72 casks.

20. A merchant finds that if he mixes sherry and brandy in quantities which are in the ratio of 2 to 1, he can sell the mixture, at 78 dimes a dozen; but if the ratio be as 7 to 2, he must sell it at 79 dimes a dozen. What is the price of each per dozen?

Ans. Sherry 81, and brandy 72 dimes per dozen.

21. Round two wheels, whose circumferences are as 5 to 3, two ropes are wrapped, whose difference exceeds the difference of the circumferences by 280 yards. Now the longer rope applied to the larger wheel wraps round it a certain number of times, greater by 12, than the shorter round the smaller wheel; and if the larger wheel turns round three times as quick as the other, the ropes will be discharged at the same time. What are the lengths of the ropes and the circumferences of the wheels?

Ans. The ropes, 360 and 72 yards; and circumferences of the wheels 20 and 12 yards.

22. If A and B together can perform a piece of work in 8 days, A and C together in 9 days, and B and C together in 10 days, in how many days can each alone perform the same work?

Ans. A in $14\frac{34}{49}$ days, B in $17\frac{23}{41}$ days, and C in $23\frac{7}{31}$ days.

23. Three brothers bought a vineyard for \$100. The youngest says, that he could pay for it alone, if the second would give him $\frac{1}{2}$ the money he had; the second says, that if the eldest would give him only the $\frac{1}{3}$ of his money, he could pay for the vineyard; lastly, the eldest asks only $\frac{1}{4}$ part of the money of the youngest to pay for the vineyard himself. How much money had each?

Ans. The oldest had \$84, the 2d \$72, and the youngest \$64.

24. Three persons, A, B, and C, play together. In the first game A loses to each of the other two, as much money as each of them had when they commenced. In the next game, B loses to each of the other two, as much money as they each had at the commencement of the 2d game. In the third game, C loses to each of the other two as much as they each had at the commencement of the 3d game. On leaving off, they find that each has an equal sum, namely, \$24. With how much money did each commence?

Ans. A \$39, B \$21, and C \$12.

25. Three laborers, A, B, and C, are employed to do a certain piece of work. A and B can do the work in a days; A and C in b days; and B and C in c days. How long would it take each to do the work alone, and how long when they all work together?

$$\text{Ans.}\begin{cases} A \text{ requires } \dfrac{2abc}{bc+ac-ab} \text{ days,} \\ B \text{ requires } \dfrac{2abc}{bc+ab-ac} \text{ days,} \\ C \text{ requires } \dfrac{2abc}{ab+ac-bc} \text{ days,} \\ A, B, \text{ and } C \text{ require } \dfrac{2abc}{ab+ac+bc} \text{ days.} \end{cases}$$

26. A cistern containing 210 buckets, may be filled by 2 pipes. By an experiment, in which the first was open 4, and the second 5 hours, 90 buckets of water were obtained. By another experiment, when the first was open 7, and the other $3\frac{1}{2}$ hours, 126 buckets were obtained. How many buckets does each pipe discharge in an hour? And in what time will the cistern be filled, when the water flows from both pipes at once?

Ans. The first pipe discharges 15, and the second, 6 buckets in an hour, and it will require 10 hours for them to fill the cistern.

27. A person has two horses, and two saddles. The better saddle cost \$50, the other \$2. If he places the better saddle upon the first horse, and the worse upon the second, then the latter is worth \$8 less than the other; but if he puts the worse saddle upon the first, and the better upon the second horse, the latter is worth $3\frac{3}{4}$ times as much as the former. What is the value of each horse?

Ans. The 1st \$30, and the 2d \$70.

28. A company at an inn, expended a certain sum of money; for the payment of which they agree to contribute equally. Had there been 5 persons more, and had each spent $12\frac{1}{2}$ cents more, then the bill would have been \$10.17$\frac{1}{2}$; but had there been 3 persons less, and had each expended 5 cents less, the bill would have been \$8.25. How many were there in the company, and what did each spend?

Ans. There were 11 persons, and each spent 80 cents.

29. A work is to be printed so that each page may contain a certain number of lines, and each line a certain number of letters. If each page should contain 3 lines more, and each line 4 letters more, then there would be 224 letters more on each page; but if there

should be 2 lines less on a page, and 3 letters less in each line, then each page would contain 145 letters less. How many lines are there on each page, and how many letters in each line?

Ans. 29 lines in a page, and 32 letters in a line.

30. A coach set out from Indianapolis to Cincinnati with a certain number of passengers; 4 more being on the outside than within. Seven outside passengers could travel at 50 cents less expense than 4 inside. The fare of the whole amounted to $45. But at the end of half the journey, it took up 3 more outside and 1 more inside passengers; in consequence of which, the whole fare was increased in the ratio of 17 to 15. What was the number of passengers, and the fare of each?

Ans. There were 5 inside, and 9 outside passengers. The fare of each inside passenger was $4.50, and of each outside passenger $2.50.

INTERPRETATION OF NEGATIVE RESULTS.

THEOREM.

(285.) When the value of an unknown quantity in an equation is found to be negative, this result indicates that an absurdity is involved in the enunciation of the problem.

DEMONSTRATION.

Find a number which, added to $+4$, makes 2.

Let $x =$ the required number.

$$\text{Then } x+4=2$$
$$x=-2$$

This result indicates that the problem was incorrectly enunciated; the words *added to* being used for *subtracted from.* The problem is, however, worded correctly, if we consider the word *added* in its extended algebraic sense. Let us word this problem differently:

Find a number to which if 4 *be added the sum will be* 2.

The result $x=-2$ indicates that the problem is absurd in an arithmetical sense, but definite in an algebraic sense.

Let us take another problem:

A father whose age is 42 years has a son whose age is 12. *In how many years will the age of the son be $\frac{1}{4}$ that of the father?*

This question gives the equation

$$x+12=\frac{42+x}{4}.$$

The solution of which gives $x=-2$.

This result shows that the son's age will never be, *in the future*, $\frac{1}{4}$ the father's age; but that 2 years *ago* the son's age was $\frac{1}{4}$ that of the father.

That part of the question which we have put in italics should have been, *How many years since the age of the son was $\frac{1}{4}$ that of the father?*

Let us take still another example.

A man worked 7 days, and had his son with him 3 days; and received for wages \$2.20. He afterward worked 5 days, and had his son with him 1 day, and received for wages \$1.80. What were the father's daily wages, and what was the effect of the son's presence?

Letting $x =$ the father's daily effect,

And $y =$ " son's " "

We get the equations,

$$\left.\begin{aligned} 7x+3y&=220, \\ \text{and } 5x+\ y&=180, \end{aligned}\right\} (1).$$

provided the daily effect of each is productive of wages.

But, if the daily effect of the father adds to the amount of wages, and the daily effect of the son diminishes the amount of wages, the equations must be

$$\left.\begin{aligned} 7x-3y&=220, \\ \text{and } 5x-\ y&=180. \end{aligned}\right\} (2).$$

If the reverse were true, we should have

$$\left.\begin{aligned} -7x+3y&=220, \\ \text{and } -5x+\ y&=180. \end{aligned}\right\} (3).$$

Which of these three couplets is the one which belongs to this problem? This question may be answered after we have found the values of x and y in each.

The (1) gives $x=40$ and $y=-20$; the (2), $x=40$ and $y=20$; and the (3), $x=-40$, $y=-20$.

Now, the (3) results can not be true, because they make the daily effect of each diminish the amount of wages, while the equations from which these values were derived considered this to be true only of the father's daily effect.

The (1) results, in like manner, indicate that the daily effect of the son is to *diminish* the amount of wages, while the equations from which they were derived considered both to be productive of wages.

The results derived from the (2) couplet are the only ones that fulfill the arithmetical conditions. The form of the (2) couplet, also, shows how the question should be stated; or, in other words, that the son's presence diminished, each day, the father's earnings by an amount equal to 20 cents.

We may suppose the father had to allow his employer 20 cents a day for his son's board; or that he was hindered one-half a day in his work every day the son was present; or that the father, out of his own wages, requested the employer to allow the son 20 cents for every day the son worked.

Guided by such ideas, we are enabled generally to give an arithmetical explanation of negative results.

QUESTIONS.

1. A man, at the time of his marriage, was 50 years old, and his wife 40. When was he twice as old as she?

Ans. 30 years before marriage.

2. What fraction is that which becomes $\frac{3}{5}$ when 1 is added to its numerator, and becomes $\frac{4}{7}$ when 1 is added to its denominator?

Ans. $\frac{-10}{-15}$.

3. A man, when he was married, was 30 years old, and his wife 15. How many years must elapse before his age will be three times his wife's age.

Ans. He was three times as old as she $7\frac{1}{2}$ years before their marriage.

4. What fraction is that which, if 2 be added to its numerator, its value is zero; but, if 6 be added to its denominator, its value is infinite?

Ans. $\frac{-2}{-6}$.

5. What fraction is that which, if 3 be added to its numerator, its value is nothing; but, if 4 be subtracted from its denominator, its value is 1?

Ans. $\frac{-3}{1}$.

6. A laborer working for a gentleman during 12 days, having with him, the first 7 days, his wife and son, who occasion an expense to

him, received \$4.60; he afterward worked 8 days, during 5 of which his wife and son were with him, and received \$3.00. What were the wages of the laborer per day, and also the expense, per day, of his wife and son?

Ans. His daily wages 50 cents, and expense of wife and son 20 cents.

7. Two men, A and B, commenced trade at the same time; A had 3 times as much money as B; A gained \$400, and B \$150; now, A has twice as much money as B. How much had each at first?

Ans. A was in *debt* \$300, and B \$100.

8. A man worked 10 days, his wife 4 days, and his son 3 days, and their wages amounted to \$11.50; at another time, he worked 9 days, his wife 8 days, and his son 6 days, and their wages amounted to \$12.00; a third time, he worked 7 days, his wife 6 days, and his son 4 days, and their wages amounted to \$9.00. What were the daily wages of each?

Ans. Husband's daily wages, \$1.00; wife's, 0; and son's, 50 cents.

9. The sum of two numbers is 120, and their difference is 160; what are the numbers? *Ans.* 140 and -20.

10. What number is that whose $\frac{1}{4}$ part exceeds its $\frac{1}{3}$ part by 12?

Ans. -144.

GENERAL DISCUSSION OF CERTAIN RESULTS OBTAINED IN THE SOLUTION OF SIMPLE EQUATIONS.

PROBLEM.

(286.) To interpret the result $0=0$.

SOLUTION.

Let us endeavor to solve the problem:

$$\text{Given } \left\{\begin{aligned} \frac{y}{b}+\frac{x}{a}&= c \ (1), \\ \frac{ay}{b}+x&=ac \ (2), \end{aligned}\right\} \text{ to find the values of } x \text{ and } y.$$

$$\frac{ay}{b}+x=ac \qquad (3)=(1)\times a$$

$$0=0 \qquad (4)=(2)-(3)$$

Hence, we are unable to obtain the values of x and y.

By looking at the second equation, we see that it is not independent of the first, it being the first multiplied by a.

We have, then, only $\frac{y}{b}+\frac{x}{a}=c$ to find the values of x and y.

We have already shown that when there are two unknown quantities and but one equation, the values of x and y are indeterminate, or, in other words, that there are an infinite number of corresponding values of x and y, which will satisfy the equation.

Again, let $2x+a=x+\frac{1}{2}a+x+\frac{1}{2}a$.

This equation, also, reduces to $0=0$. This ought to be the case, because $2x+a=2x+a$, or $2x=2x$, or $x=x$, is an indeterminate equation, since x may be any quantity whatever.

Hence, the result $0=0$ *is a sign of indetermination, and when obtained from two simultaneous equations, indicates that there is but one condition.* When the values of x and y are indeterminate, it is customary to indicate it by $x=\frac{0}{0}$ and $y=\frac{0}{0}$.

PROBLEM.

(287.) Find a number such, that if 9 be added to 8 times the number, and this sum be divided by 6, the quotient will be equal to 13 augmented by 12 times the number, and this sum divided by 9.

SOLUTION.

Let x = the required number.

Then we have, by the conditions given,

$$\frac{8x+9}{6}=\frac{13+12x}{9} \qquad (1)$$

$$72x+81=78+72x \qquad (2)=(1)\times 54.$$

$$72x-72x=78-81$$

$$81-78=72x-72x$$

$$3=0$$

This result is an *absurdity*, and indicates that the conditions of the question are *incompatible* or *contradictory*.

The conditions, we have seen, give

$$72x+81=72x+78.$$

This may assume the form

$$72x+78+3=72x+78.$$

This equation can be satisfied only on the condition $3=0$; but 3 does not equal 0, and therefore the equation is impossible.

PROBLEM.

(288.) To interpret $x=\frac{\infty}{\infty}$.

SOLUTION.

We may consider that $\infty=\frac{1}{a}$ when $a=0$, and, also, $\infty=\frac{1}{b}$ when $b=0$. Then, $x=\frac{\frac{1}{a}}{\frac{1}{b}}$, becomes $x=\frac{\infty}{\infty}$, when both a and b are both equal to 0.

But $x=\frac{\frac{1}{a}}{\frac{1}{b}}=\frac{1}{a}\div\frac{1}{b}=\frac{1}{a}\times\frac{b}{1}=\frac{b}{a}$.

Now, if we make a and b each equal to 0, we have $x=\frac{0}{0}$.

We see by this that we can get $x=\frac{\infty}{\infty}$ and $x=\frac{0}{0}$ from the same equation, $x=\frac{\frac{1}{a}}{\frac{1}{b}}$, by making both a and b equal 0;

Hence, $\frac{\infty}{\infty}$ is equivalent to $\frac{0}{0}$, or, in other words, is *sometimes* a symbol of indetermination, for we have already shown that $\frac{0}{0}$ is sometimes a sign of indetermination. In vanishing fractions the value of $\frac{0}{0}$ is determinate.

PROBLEM.

(289.) To interpret $x=0\,.\,\infty$.

SOLUTION.

Letting $x=a\times\frac{1}{b}$, and making both a and b equal to 0, we get $x=0\times\frac{1}{0}$, or $x=0\,.\,\infty$.

But $x=a\times\frac{1}{b}$ is the same as $x=\frac{a}{b}$, or, when a and b are both equal to 0, $x=\frac{0}{0}$.

We see, then, that $0 \cdot \infty$ is equivalent to $\frac{0}{0}$.

PROBLEM.

(290.) To interpret $x = \infty - \infty$.

SOLUTION.

Let $x = \frac{1}{a} - \frac{1}{b}$.

If, in this equation, both a and b are 0, we have

$$x = \frac{1}{0} - \frac{1}{0}, \text{ or } x = \infty - \infty.$$

But $x = \frac{1}{a} - \frac{1}{b}$ is the same as

$$x = \frac{b-a}{ab}.$$

Which equation becomes, when a and b are each equal to 0,

$$x = \frac{0-0}{0 \cdot 0},$$

$$\text{or, } x = \frac{0}{0}.$$

We see, then, that $\infty - \infty$ is equivalent to $\frac{0}{0}$.

PROBLEM.

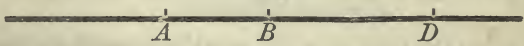
(291.) Two couriers traveled on the same road; when one was at A the other was at B. When were they together?

SOLUTION.

Let 

represent the road and the given places. This problem is stated in very general language, and, therefore, embodies many conditions. It will, however be found not to be so general as the algebraic formula to which it gives rise.

To obtain this formula let us take one of the many cases which may occur, viz., that the faster traveler was at A when the other was at B, and that they were together at D.



Let a = the distance from A to B; m = the number of miles the faster traveler went per hour; and n = the number of miles the other went per hour.

Let $x=AD$, and $y=BD$.

We have then $\frac{x}{m}$ the number of hours that it took the faster courier to go the distance x, and $\frac{y}{n}=$ the number of hours it took the other to go the distance y. Since these terms must be equal, we have

$$\frac{x}{m}=\frac{y}{n}$$

$$\text{But } x-y=a$$

The solution of these equations give

$$x=\frac{am}{m-n}$$

$$\text{and } y=\frac{an}{m-n}$$

Whence, $\frac{x}{m}$, or $\frac{y}{n}=\frac{a}{m-n}$

These results are applicable to a great number of separate suppositions, or we may consider them as the solution of the general problem.

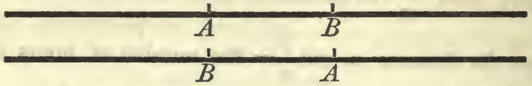
In order to adapt these results to all the supposable cases, we must first establish certain conventional signs.

When m is a positive quantity, we shall consider that the traveler who went m miles per hour, went eastward, or to the right; but when m is a negative quantity, we shall consider that he went westward, or to the left. We shall, also, attach the same ideas to n. When x is a positive quantity, we shall consider that D is the eastward, or to the right of A; but when it is a negative quantity, we shall consider that D is westward, or to the left of A. Also, we shall consider D as eastward or westward from B, according as y is positive or negative.

When $\frac{x}{m}$ or $\frac{y}{n}$ is positive, we shall consider the travelers were together after they were at A and B, but when $\frac{x}{m}$ or $\frac{y}{n}$ is negative, that they were together before they were at A and B. But how shall we establish the meaning of $+a$ and $-a$, since, according to the above ideas, the distance a would have a different sign according as we refer it to A or B?

We shall let the point A rule; and shall call AB, or a, positive

when B is to the right of A; and AB, or a, negative when B is to the left of A. Thus



We shall consider AB in the first equal to $+a$, but in the second line equal to $-a$.

With conventional ideas, we can make a particular problem out of each of the following suppositions, and the corresponding values deduced from the above formulas will be the correct results for each particular supposition.

1st. When $m=+2$, $n=+1$, and $a=+3$, we get $x=+6$, and $y=+3$, $\frac{x}{m}=+3$, and $\frac{y}{n}=+3$.

2d. When $m=-2$, $n=-1$, and $a=+3$, we get $x=+6$, $y=+3$, $\frac{x}{m}=-3$, and $\frac{y}{n}=-3$.

3d. When $m=+2$, $n=-1$, and $a=+3$, we get $x=+2$, $y=-1$, $\frac{x}{m}=+1$, and $\frac{y}{n}=+1$.

4th. When $m=-2$, $n=+1$, and $a=+3$, we get $x=+2$, $y=-1$, $\frac{x}{m}=-1$, and $\frac{y}{n}=-1$.

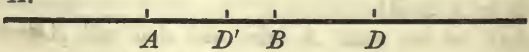
5th. When $m=+2$, $n=+1$, and $a=-3$, we get $x=-6$, $y=-3$, $\frac{x}{m}=-3$, and $\frac{y}{n}=-3$.

6th. When $m=-2$, $n=-1$, and $a=-3$, we get $x=-6$, $y=-3$, $\frac{x}{m}=+3$, and $\frac{y}{n}=+3$.

7th. When $m=+2$, $n=-1$, and $a=-3$, we get $x=-2$, $y=+1$, $\frac{x}{m}=-1$, and $\frac{y}{n}=-1$.

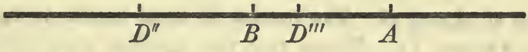
8th. When $m=-2$, $n=+1$, and $a=-3$, we get $x=-2$, $y=+1$, $\frac{x}{m}=+1$, and $\frac{y}{n}=+1$.

In the first four suppositions B is east of A, and in the last four, B is west of A.



In the first two suppositions the couriers were together at D, in the

1st 3 hours *after* they were at A and B, and in the 2d 3 *before.* In the next two suppositions they were together at D', in the 3d 1 hour *after* they were at A and B, and in the 4th 1 hour *before.*



In the next two suppositions, they were together at D'', in the 5th 3 hours *before* they were at A and B, and in the 6th 3 hours *after.* In the next two suppositions they were together at D''', in the 7th 1 hour *before* they were at A and B, and in the 8th, 1 hour *after.*

9th. When $m=+c$, $n=+c$, and $a=+d$, we get $x=+\infty$, $y=+\infty$, $\frac{x}{m}=+\infty$, and $\frac{y}{n}=+\infty$.

10th. When $m=-c$, $n=-c$, and $a=+d$, we get $x=+\infty$, $y=+\infty$, $\frac{x}{m}=-\infty$, and $\frac{y}{n}=-\infty$.

11th. When $m=+c$, $n=-c$, and $a=+d$, we get $x=+\frac{d}{2}$, $y=-\frac{d}{2}$, $\frac{x}{m}=+\frac{d}{2c}$, and $\frac{y}{n}=+\frac{d}{2c}$.

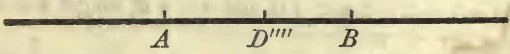
12th. When $m=-c$, $n=+c$, and $a=+d$, we get $x=+\frac{d}{2}$, $y=-\frac{d}{2}$, $\frac{x}{m}=-\frac{d}{2c}$, and $\frac{y}{n}=-\frac{d}{2c}$.

13th. When $m=+c$, $n=+c$, and $a=-d$, we get $x=-\infty$, $y=-\infty$, $\frac{x}{m}=-\infty$, and $\frac{y}{n}=-\infty$.

14th. When $m=-c$, $n=-c$, and $a=-d$, we get $x=-\infty$, $y=-\infty$, $\frac{x}{m}=+\infty$, and $\frac{y}{n}=+\infty$.

15th. When $m=+c$, $n=-c$, and $a=-d$, we get $x=-\frac{d}{2}$, $y=+\frac{d}{2}$, $\frac{x}{m}=-\frac{d}{2c}$, and $\frac{y}{n}=-\frac{d}{2c}$.

16th. When $m=-c$, $n=+c$, and $a=-d$, we get $x=-\frac{d}{2}$, $y=+\frac{d}{2}$, $\frac{x}{m}=+\frac{d}{2c}$, and $\frac{y}{n}=+\frac{d}{2c}$.



In the 9th supposition the couriers will be together at a point in-

finitely distant to the east of A and B in an infinite number of hours from the time they were at A and B. In the 10th supposition they were together at the same place an infinite number of hours before they were at A and B. The reverse of these results is found in suppositions 13 and 14. To say that two couriers were together an infinite number of hours ago, is the same as saying they never were together, and to say they will be together in an infinite number of hours, is the same as saying they never will be together.

It should be remembered that in the 13th and 14th that A is east of B.

It will be seen that to obtain these infinite results, we have considered 0 at one time as $+0$, and at another -0. We started out with the idea that the position A, or that of the faster traveler, should rule the sign of a. But when both travel equally fast, there seems to be a difficulty. To remove this, we consider m disregarding its sign as greater than n by an infinitely small quantity. Hence, $m-n=+0$ and $-m+n=-0$.

In suppositions 11, 12, 15, and 16, the couriers were together at D'''', a point midway between A and B, A being the east point in 11 and 12, and the west point in 14 and 15. In these four suppositions, the number of hours was the same, $\frac{d}{2c}$ hours. $+\frac{d}{2c}$ indicating that they were together $\frac{d}{2c}$ hours *after* they were at A and B, and $-\frac{d}{2c}$ that they were together $\frac{d}{2c}$ hours *before* they were at A and B.

17th. When $m=+c$, $n=+0$, and $a=+d$, we get $x=+d$, $y=+0$, $\frac{x}{m}=+\frac{d}{c}$, and $\frac{y}{n}=+\frac{0}{0}$.

18th. When $m=-c$, $n=-0$, and $a=+d$, we get $x=+d$, $y=+0$, $\frac{x}{m}=-\frac{d}{c}$, and $\frac{y}{n}=-\frac{0}{0}$.

19th. When $m=+c$, $n=-0$, and $a=+d$, we get $x=+d$, $y=-0$, $\frac{x}{m}=+\frac{d}{c}$, and $\frac{y}{n}=+\frac{0}{0}$.

20th. When $m=-c$, $n=+0$, and $a=+d$, we get $x=+d$, $y=-0$, $\frac{x}{m}=-\frac{d}{c}$, and $\frac{y}{n}=-\frac{0}{0}$.

21st. When $m=+c$, $n=+0$, and $a=-d$, we get $x=-d$, $y=-0$, $\frac{x}{m}=-\frac{d}{c}$, and $\frac{y}{n}=-\frac{0}{0}$.

22d. When $m=-c$, $n=-0$, and $a=-d$, we get $x=-d$, $y=-0$, $\frac{x}{m}=+\frac{d}{c}$, and $\frac{y}{n}=+\frac{0}{0}$.

23d. When $m=+c$, $n=-0$, and $a=-d$, we get $y=-d$, $y=+0$, $\frac{x}{m}=-\frac{d}{c}$, and $\frac{y}{n}=-\frac{0}{0}$.

24th. When $m=-c$, $n=+0$, and $a=-d$, we get $x=-d$, $y=+0$, $\frac{x}{m}=+\frac{d}{c}$, and $\frac{y}{n}=+\frac{0}{0}$.

In these results we get $\frac{y}{n}=+\frac{0}{0}$, and $\frac{y}{n}=-\frac{0}{0}$. But $\frac{y}{n}$ is not therefore indeterminate, because $\frac{y}{n}=\frac{x}{m}$, and $\frac{x}{m}$ is either $+\frac{d}{c}$ or $-\frac{d}{c}$.

$n=+0$ indicates that the courier who was at B stood still with his face to the east, and $n=-0$ that he stood still with his face to the west.

25th. When $m=+0$, $n=+0$, and $a=+d$, we get $x=+\frac{0}{0}$, $y=+\frac{0}{0}$,

$\frac{x}{m}=+\frac{\frac{0}{0}}{0}$, and $\frac{y}{n}=\frac{\frac{0}{0}}{0}$.

26th When $m=-0$, $n=-0$, and $a=+d$, we get $x=+\frac{0}{0}$, $y=+\frac{0}{0}$,

$\frac{x}{m}=-\frac{\frac{0}{0}}{0}$, and $\frac{y}{n}=-\frac{\frac{0}{0}}{0}$.

27th. When $m=+0$, $n=-0$, and $a=+d$, we get $x=+\frac{0}{0}$, $y=-\frac{0}{0}$,

$\frac{x}{m}=+\frac{\frac{0}{0}}{0}$, and $\frac{y}{n}=+\frac{\frac{0}{0}}{0}$.

28th. When $m=-0$, $n=+0$, and $a=+d$, we get $x=+\frac{0}{0}$, $y=-\frac{0}{0}$,

$\frac{x}{m}=-\frac{\frac{0}{0}}{0}$, and $\frac{y}{n}=-\frac{\frac{0}{0}}{0}$.

29th. When $m=+0$, $n=+0$, and $a=-d$, we get $x=-\frac{0}{0}$, $y=-\frac{0}{0}$, $\frac{x}{m}=-\frac{\frac{0}{0}}{0}$, and $\frac{y}{n}=-\frac{\frac{0}{0}}{0}$.

30th. When $m=-0$, $n=-0$, and $a=-d$, we get $x=-\frac{0}{0}$, $y=-\frac{0}{0}$, $\frac{x}{m}=+\frac{\frac{0}{0}}{0}$, and $\frac{y}{n}=+\frac{\frac{0}{0}}{0}$.

31st. When $m=+0$, $n=-0$, and $a=-d$, we get $x=-\frac{0}{0}$, $y=+\frac{0}{0}$, $\frac{x}{m}=-\frac{\frac{0}{0}}{0}$, and $\frac{y}{n}=-\frac{\frac{0}{0}}{0}$.

32d. When $m=-0$, $n=+0$, and $a=-d$, we get $x=-\frac{0}{0}$, $y=+\frac{0}{0}$, $\frac{x}{m}=+\frac{\frac{0}{0}}{0}$, and $\frac{y}{n}=+\frac{\frac{0}{0}}{0}$.

These results are peculiar, and are not indeterminate in the sense usually attached to the word indeterminate, since its technical meaning is that there is an infinite number of values which will satisfy the given conditions. But there is an absurdity in asking when they were together if there was a distance between them and each stood still. We should obtain the same results if at the same time a was made $+0$ or -0, and then they would be the true symbols of indetermination.

If, in the first eight suppositions, we put $+0$ instead of $+3$, and -0 instead of -3, we get $x=+0$ or $x=-0$, $y=+0$ or $y=-0$, $\frac{x}{m}=+0$ or $\frac{x}{m}=-0$, which results are easily interpreted.

If, in the second eight suppositions, we put $a=+0$ or $a=-0$, we get $x=+\frac{0}{0}$ or $x=-\frac{0}{0}$, $y=+\frac{0}{0}$, or $y=-\frac{0}{0}$, and $\frac{x}{m}=+\frac{0}{0}$, or $\frac{x}{m}=-\frac{0}{0}$, which are true symbols of indetermination, showing that they were always together and always will be.

If, in the third eight suppositions, we put $a=+0$ or $a=-0$, we

get $x=+0$ or $x=-0$, $y=+0$ or $y=-0$, $\frac{x}{m}=+0$ or $\frac{x}{m}=-0$, and $\frac{y}{n}=+\frac{0}{0}$ or $\frac{y}{n}=-\frac{0}{0}$, which results are easily interpreted.

We have then made 64 suppositions, all of which admit of a simple interpretation except eight, which embodied an absurdity.

These suppositions and results show that algebraic formulas are far more extensive than the particular problem from which they may be derived.

EXAMPLES.

1. Given $\left\{\begin{array}{l} 6x+9y=27, \\ 8x+12y=36, \end{array}\right\}$ to find the values of x and y.

Ans. $x=\frac{0}{0}$, $y=\frac{0}{0}$.

2. Given $\left\{\begin{array}{l} 6x+3y=3, \\ 4x+2y=2, \end{array}\right\}$ to find the values of x and y.

Ans. $x=\frac{0}{0}$, $y=\frac{0}{0}$.

3. Given $\left\{\begin{array}{l} 2x-y=1, \\ 5x-2y=4, \\ 3x+y=9, \\ 3x-y=2, \end{array}\right\}$ to find the values of x and y.

Ans. No value can be found that will at the same time satisfy all the equations.

4. Given $\left\{\begin{array}{l} 2x-y=1, \\ 5x-2y=4, \\ 3x+y=9, \\ 3x-y=3, \end{array}\right\}$ to find the values of x and y.

Ans. $x=2$, $y=3$. Explain these results.

5. Given $\left\{\begin{array}{l} x+y+z=3, \\ x-y-z=1, \\ 2x+2y+2z=1, \end{array}\right\}$ to find the values of x and y.

Ans. $x=\frac{0}{0}$, $y=-\infty$, $z=+\infty$.

6. Given $\left\{\begin{array}{l} x+y+2z=2, \\ x+y+2z=1, \\ x+y+2z=-1, \end{array}\right\}$ to find the values of x, y, and z.

Ans. $x=\frac{0}{0}$, $y=\frac{0}{0}$, $z=\frac{0}{0}$.

NOTE.—The results in the last two examples were obtained by the checker-board process, and show that $\frac{0}{0}$ is not always a symbol of indetermination, for in these examples the equations are incompatible.

SIMPLE INEQUATIONS.

(292.) An *inequation* is that which denotes that one algebraic expression is greater or less than another. Thus $a>b$, and $c<d$ are inequations, which denote that a is greater than b, and that c is less than d.

(293.) Two inequations subsist in the same sense when the sign of inequality has the same position in both. Thus $a>b$ and $c>d$ are inequations in the same sense; so also, are $a<b$ and $c<d$.

(294.) Two inequations subsist in a contrary sense when the sign of inequality has not the same position in both. Thus $a>b$ and $c<d$ are inequations in a contrary sense; so also, are $a<b$ and $c>d$.

THEOREM.

(295.) The addition or subtraction of an equation to or from an inequation results in an inequation in the same sense.

THEOREM.

(296.) The addition of two inequations which subsist in the same sense, results in an equation in the same sense.

THEOREM.

(297.) The subtraction of an inequation from another which exists in the same sense, does not necessarily result in an inequation in the same sense.

DEMONSTRATION.

Subtracting $4>3$ from $8>5$, we get $4>2$ which is true.

But if we subtract $8>5$ from $4>3$, we get $-4>-2$, which is not true according to the conventional idea sometimes attached to negative quantities, that they are less than nothing.

But this inequation is still true if we limit the reasoning to negative quantities, for -4 is evidently a greater negative quantity than -2. A debt of 4 dollars is greater than a debt of 2 dollars.

On the other hand, we say that a man who is in debt 2 dollars, and has nothing, is worth more than the man who is in debt 4 dollars and has nothing.

Hence considered in the light of wealth -2 is greater than -4, or in other words, -2 comes nearer being a positive quantity than -4 does. Again, let us subtract $8>2$ from $11>9$, and we have $3<7$, an inequation which subsists in a contrary sense. Let us put these inequations in a different form:

$$9+2>9$$
$$\text{and}\quad 2+6>2$$

Subtracting we get $-4>0$, dropping equal quantities from both sides. But we have just seen that this inequation must subsist in a contrary sense, that is $-4<0$. This shows -4 must be considered less than nothing.

We can also prove this as follows:

Since $3<7$ it follows that

$$3-7<7-7$$
$$-4<0$$

THEOREM.

(298.) The multiplication or division of an inequation by a positive quantity, results in an inequation in the same sense.

THEOREM.

(299.) The multiplication or division of an inequation by a negative quantity, results in an inequation in a contrary sense.

DEMONSTRATION.

Let $a>b$. Now if we multiply this by $-c$, we get $-ac<-bc$, because the greater a negative quantity is numerically the less it is considered.

Thus, $4<5$ gives, by multiplying by -6, $-24>-30$.

Since, multiplying by -1 is equivalent to changing signs, we derive the

COROLLARY.

When the signs in both members of an inequation are changed, the sign of inequality must be reversed.

Thus changing the signs in $-a+b-c>d-m$, we get

$$a-b+c<m-d.$$

THEOREM.

(300.) When both members of an inequation are positive, they may be raised to the same power, and the result will be an inequation existing in the same sense.

THEOREM.

(301.) When both members of an inequation are positive, the same root of each may be extracted and the result will be an inequation existing in the same sense.

PROBLEM

(302.) Find the limit to the value of x in the inequation

$$7x-\frac{23}{3}>\frac{2x}{3}+5 \quad (1).$$

SOLUTION.

$$\begin{array}{rl} 21x-23>2x+15 & (2)=(1)\times 3 \\ 19x>38 & (3)=(2) \text{ transposed.} \\ x>2 & (4)=(3)\div 19. \end{array}$$

EXAMPLES.

1. Given $x+\frac{1}{2}x+\frac{1}{3}x>11$ to find the limit of x. *Ans.* $x>6$.

2. Given $3x+7x-30>10$ to find the limit of x. *Ans.* $x>4$.

3. Given $\frac{a^2-b^2}{6d}>\frac{a^2-b^2}{3x}$ to find the limit of x. *Ans.* $x>2d$.

4. Given $\frac{1}{2}x+3x-5>16$ to find the limit of x. *Ans.* $x>6$.

5. Given $7x-1>34$ to find the limit of x. *Ans.* $x>5$.

6. Given $\left\{\begin{array}{l} 4x-6<2x+4, \\ 2x+4>16-2x, \end{array}\right\}$ to find an integer value of x.

Ans. $x=4$.

7. Given $\left\{\begin{array}{l} \frac{ax}{5}+bx-ab>\frac{a^2}{5}, \\ \frac{bx}{7}-ax+ab<\frac{b^2}{7}, \end{array}\right\}$ to find the limit of x.

Ans. $x>a$ and $x<b$.

8. Prove that a^2+b^2 is equal to, or greater than $2ab$, according as a and b are equal or unequal.

9. Prove that a^3+1 is equal to, or greater than a^2+a, according as a is equal to 1, or is a positive quantity, that differs from 1.

10. Prove that $a^3+1<a^2+a$ when a is a negative integer, or an improper fraction, that differs from -1.

11. Prove that $a^3+1>a^2+a$ when a is a negative proper fraction.

12. Prove that $\frac{a}{b}+\frac{b}{a}>2$ when a and b are both positive or both negative.

13. Prove that $\frac{a}{b}+\frac{b}{a}<2$ when a and b are not both positive or both negative.

14. Prove that $a-b>(\sqrt{a}-\sqrt{b})^2$ when $a>b$, and both are considered positive.

15. Prove that $\frac{a^3+b^3}{a^2+b^2}>\frac{a^2+b^2}{a+b}$ when a and b are unequal.

16. Prove that $\frac{x}{y^2}+\frac{y}{x^2}>\frac{1}{x}+\frac{1}{y}$ when x and y are unequal.

17. Given $\left\{\begin{array}{l} x^2=a^2+b^2 \\ y^2=c^2+d^2 \end{array}\right\}$ to find the limit of xy.

Ans. $xy>ac+bd$.

18. Prove that $a^2+b^2+c^2>ab+ac+bc$ when a, b, and c are not all equal.

19. Prove that $\sqrt{3}+3\sqrt{2}>\sqrt{6}+\sqrt{12}$.

20. Prove that $\frac{a+c+e}{b+d+f}>\frac{e}{f}$ and $<\frac{a}{b}$, $\frac{e}{f}$ being the least, and $\frac{a}{b}$ the greatest of the fractions, $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$.

21. Prove that $xyz>(x+y-z)(x+z-y)(y+z-x)$ when x, y, and z are not all equal.

22. A shepherd being asked the number of his sheep, replied, that double their number diminished by 7 is greater than 29, and triple their number diminished by 5 is less than double their number increased by 16. What was the number of his sheep?

Ans. 19 or 20.

23. A market woman has a number of oranges, such, that triple the number increased by 2, exceeds double the number increased by 61; and 5 times the number diminished by 70, is less than 4 times the number diminished by 9. How many oranges has she?

Ans. 60.

24. The sum of two whole numbers is 25; if the greater be divided by the less, the quotient will be greater than 3; and if the less be divided by the greater, the quotient will be greater than $\frac{1}{5}$. What are the numbers? *Ans.* 20 and 5.

25. What whole number is that which, if doubled and diminished by 6, is greater than 24; but, if tripled and diminished by 6, is less than double the number increased by 10?

Ans. There is no such whole number.

26. The sum of two whole numbers is 32, and, if the greater be divided by the less, the quotient will be less than 5, but greater than 2. What are the numbers? *Ans.* 24 and 8.

27. Twice a certain number, increased by 7, is not greater than 19; and thrice the same number, diminished by 5, is not less than 13. What is the only number, whether whole or fractional, that will satisfy these conditions? *Ans.* 6.

28. Four, added to five times a whole number, is greater than 19, added to twice the number; and 4, subtracted from five times the number is less than 4 added to 4 times the number. What is the number? *Ans.* 6 or 7.

29. Given $\left\{\begin{matrix} 3x-4 < x+6, \\ 5x+7 > 3x+13, \end{matrix}\right\}$ to find x in whole numbers.

Ans. $x=4$.

30. Given $\left\{\begin{matrix} \frac{1}{4}(x+2)+\frac{1}{3}x < \frac{1}{2}(x-4)+3, \\ \frac{1}{4}(x+2)+\frac{1}{3}x > \frac{1}{2}(x+1)+\frac{1}{3}, \end{matrix}\right\}$ to find x in whole numbers. *Ans.* $x=5$.

$\frac{1}{2}(x+2) + \frac{1}{3}x = \frac{10}{3}$

$\frac{x}{2} \times 4 + \frac{1}{3}x = \frac{10}{3}$

$3x + 24 + 2x =$

CHAPTER XI.

QUADRATIC EQUATIONS.*

(303.) *Quadratic Equations* are divided into *Pure Quadratics* and *Affected Quadratics.*

PURE QUADRATICS.

(304.) A *pure quadratic equation* is one in which the unknown quantity appears in but one term, and is affected by the exponent 2, or by a fractional exponent, which, when reduced to its lowest terms, has 2 for its numerator; as, $x^2=9$, $x^{\frac{2}{3}}=4$, and $x^{\frac{4}{10}}=16$.

(305.) Every pure quadratic equation can be reduced to the form $x^2=a^2$, in which a^2 may represent any quantity, whether real or imaginary, positive or negative. Thus, $x^{\frac{2}{3}}=4$ is the same as $(x^{\frac{1}{3}})^2=4$, which becomes $(x)^2=4$, or $x^2=4$, by putting x for $x^{\frac{1}{3}}$.

PROBLEM.

(306.) To solve a pure quadratic equation.

SOLUTION.

Since every pure quadratic equation can be reduced to $x^2=a^2$, we have only to find the solution of this equation.

Taking the square root of both members, we have $x=\pm a$.

We do not place the double sign before x, because we seek only the *plus* value of x.

ANOTHER SOLUTION.

By transposition, $x^2=a^2$ becomes $x^2-a^2=0$, which, being factored, gives $(x-a)(x+a)=0$.

It is evident that this equation will be satisfied by putting either factor equal to *zero.*

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* Quadratic equations are also called equations of the SECOND DEGREE.

$$\therefore x-a=0$$
$$\text{or} \quad x+a=0$$

which being solved, give $x=a$ and $x=-a$.

SCHOLIUM.—These solutions show that a pure quadratic equation may be satisfied by substituting for the unknown quantity two values which are numerically equal but of opposite signs.

PROBLEM.

(307.) Given $x^2-17=130-2x^2$ to find the values of x.

SOLUTION.

$$x^2-17=130-2x^2,$$
$$3x^2=147,$$
$$x^2=49,$$
$$x=\pm 7.$$

EXAMPLES.

1. Given $4x^2+5=x^2+8$ to find the values of x. *Ans.* $x=\pm 1$.

2. Given $3x^2+3=x^2+6$ to find the values of x.

Ans. $x=\pm\frac{1}{2}\sqrt{6}$.

3. Given $x^2+ab=5x^2$ to find the values of x. *Ans.* $x=\pm\frac{1}{2}\sqrt{ab}$.

4. Given $ax^2+d=bx^2+c$ to find the values of x.

Ans. $x=\pm\sqrt{\dfrac{c-d}{a-b}}$.

5. Given $\dfrac{x^2}{4}+\dfrac{2}{3}=\dfrac{x^2}{8}+\dfrac{3}{2}$ to find the values of x.

Ans. $x=\pm\frac{2}{3}\sqrt{15}$.

6. Given $\dfrac{2x}{3}=\dfrac{x^2+3}{2x}$ to find the values of x. *Ans.* $x=\pm 3$.

7. Given $4x^2-8x^0=1$ to find the values of x. *Ans.* $x=\pm\frac{3}{2}$.

8. Given $3x^2-4=\dfrac{x^2+2}{5x^0}$ to find the values of x.

Ans. $x=\pm\frac{1}{7}\sqrt{77}$.

PROBLEM

(308.) 1. Given $x+\sqrt{a^2+x^2}=\dfrac{2a^2}{\sqrt{a^2+x^2}}$ (1) to find the values of x.

SOLUTION.

$$x\sqrt{a^2+x^2}+a^2+x^2=2a^2 \qquad (2)=(1)\times\sqrt{a^2+x^2}.$$

$$x\sqrt{a^2+x^2}=a^2-x^2 \qquad (3)=(2) \text{ transposed.}$$

$$a^2x^2+x^4=a^4-2a^2x^2+x^4 \qquad (4)=(3)^2.$$

$$3a^2x^2=a^4$$

$$3x^2=a^2$$

$$9x^2=3a^2 \qquad (7).$$

$$3x=\pm a\sqrt{3} \qquad (8)=\sqrt{(7)}.$$

$$x=\pm\tfrac{1}{3}a\sqrt{3}.$$

PROBLEM

2. Given $\sqrt{\frac{a^2}{x^2}+b^2}-\sqrt{\frac{a^2}{x^2}-b^2}=b$ (1) to find the values of x.

SOLUTION.

$$\sqrt{\frac{a^2}{x^2}+b^2}=\sqrt{\frac{a^2}{x^2}-b^2}+b \qquad (2)=(1) \text{ transposed.}$$

$$\frac{a^2}{x^2}+b^2=\frac{a^2}{x^2}-b^2+2b\sqrt{\frac{a^2}{x^2}-b^2}+b^2 \qquad (3)=(2)^2.$$

$$b^2=2b\sqrt{\frac{a^2}{x^2}-b^2} \qquad (4)=(3) \text{ transposed.}$$

$$\frac{b}{2}=\sqrt{\frac{a^2}{x^2}-b^2} \qquad (5)=(4)\div 2b.$$

$$\frac{b^2}{4}=\frac{a^2}{x^2}-b^2 \qquad (6)=(5)^2.$$

$$\frac{5b^2}{4}=\frac{a^2}{x^2} \qquad (7)=(6) \text{ transposed, \&c.}$$

$$x^2=\frac{4a^2}{5b^2} \qquad (8)=(7)\times\frac{4x^2}{5b^2}$$

$$x^2=\frac{4a^2\times 5}{25b^2} \qquad (9).$$

$$x=\pm\frac{2a}{5b}\sqrt{5} \qquad (10)=\sqrt{(9)}.$$

EXAMPLES.

1. Given $\frac{a}{x}+\frac{\sqrt{a^2-x^2}}{x}=\frac{x}{b}$ to find x. *Ans.* $x=\pm\sqrt{2ab-b^2}$.

2. Given $\frac{x}{\sqrt{a^2+x^2}-x}=b$ to find x. *Ans.* $x=\pm\frac{ab}{2b+1}\sqrt{2b+1}$.

3. Given $\sqrt{\frac{x}{4}+3}-\sqrt{\frac{x}{4}-3}=\sqrt{\frac{2x}{3}}$ to find x. *Ans.* $x=\pm 9\sqrt{2}$.

4. Given $\sqrt{\frac{1}{2}x+2}-\sqrt{\frac{1}{2}x-2}=\sqrt{x+3}-\sqrt{x-3}$ to find x.

Ans. $x=\pm 5$.

5. Given $\frac{1}{x+\sqrt{2-x^2}}+\frac{1}{x-\sqrt{2-x^2}}=ax$ to find x.

Ans. $x=\pm\frac{1}{a}\sqrt{a(a+1.)}$

6. Given $\frac{1}{1-\sqrt{1-x^2}}-\frac{1}{1+\sqrt{1-x^2}}=\frac{\sqrt{3}}{x^2}$ to find x. *Ans.* $x=\pm\frac{1}{2}$.

7. Given $(x+a)^{\frac{1}{2}}=\frac{a+b}{(x-a)^{\frac{1}{2}}}$ to find x.

Ans. $x=\pm(2a^2+2ab+b^2)^{\frac{1}{2}}$.

8. Given $\frac{a-\sqrt{a^2-x^2}}{a+\sqrt{a^2-x^2}}=b$ to find x. *Ans.* $x=\pm\frac{2a\sqrt{b}}{b+1}$.

9. Given $\frac{\sqrt{a^2-x^2}-\sqrt{b^2+x^2}}{\sqrt{a^2-x^2}+\sqrt{b^2+x^2}}=\frac{c}{d}$ to find x.

Ans. $x=\pm\sqrt{\frac{a^2(d-c)^2-b^2(d+c)^2}{2(d^2+c^2)}}$.

10. Given $\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{x}}=\sqrt{\frac{x}{b}}$ to find x. *Ans.* $x=\pm 2\sqrt{ab-b^2}$.

SIMULTANEOUS EQUATIONS.

PROBLEM

(309.) 1. Given $\left\{\begin{array}{l} x+y:x-y::a:b, \\ xy=c^2, \end{array}\right\}$ to find the values of x and y.

SOLUTION.

From the proportion, we get $ax-ay=bx+by$ which becomes, by substituting $\frac{c^2}{x}$ for y as obtained from $xy=c^2$,

$$ax-\frac{ac^2}{x}=bx+\frac{bc^2}{x} \qquad (1).$$

$$(a-b)x^2=c^2(a+b) \qquad (2)=(1)\times x, \text{ \&c.}$$

$$x^2=\frac{c^2(a+b)}{a-b} \qquad (3)=(2)\div(a-b).$$

$$x=\pm c\sqrt{\frac{a+b}{a-b}} \qquad (4)=\sqrt{(3)}.$$

PROBLEM

2. Given $\begin{cases} x^2-xy=56 \ (1), \\ xy-y^2=18 \ (2), \end{cases}$ to find the values of x and y.

SOLUTION.

This problem may be solved without reducing the equations to the general form of pure quadratics.

$$x^2-2xy+y^2=36 \qquad (3)=(1)-(2).$$
$$x-y=\pm 6 \qquad (4)=\sqrt{(3)}.$$
$$x=\pm 9 \qquad (5)=(1)\div(4).$$
$$y=\pm 3 \qquad (6)=(2)\div(4).$$

EXAMPLES.

1. Given $\begin{cases} x+y:y::3:1, \\ xy=18, \end{cases}$ to find the values of x and y.

Ans. $x=\pm 6$, and $y=\pm 3$.

2. Given $\begin{cases} x-y:y::4:5, \\ x^2+4y^2=181. \end{cases}$ to find the values of x and y.

Ans. $x=\pm 9$, and $y=\pm 5$.

3. Given $\begin{cases} x^4-y^4=369, \\ x^2-y^2=9, \end{cases}$ to find the values of x and y.

Ans. $x=\pm 5$, and $y=\pm 4$.

4. Given $\begin{cases} x+y:x::7:5, \\ xy+y^2=126 \end{cases}$ to find the values of x and y.

Ans. $x=\pm 15$, and $y=\pm 6$.

5. Given $\begin{cases} ax^2+bxy=c^2, \\ x-y:x::m:n, \end{cases}$ to find the values of x and y.

Ans. $\begin{cases} x=\pm c\sqrt{\dfrac{nc}{na+nb-mb}}, \\ y=\dfrac{c(n-m)}{n}\sqrt{\dfrac{nc}{na+nb-mb}}. \end{cases}$

6. Given $\begin{cases} x+y:x::5:3, \\ xy=6, \end{cases}$ to find the values of x and y.

Ans. $x=\pm 3$, and $y=\pm 2$.

7. Given $\begin{cases} x^2+xy=12, \\ y^2+xy=24, \end{cases}$ to find the values of x and y.

Ans. $x=\pm 2$, and $y=\pm 4$.

8. Given $\begin{cases} x^2+y\sqrt{xy}=9, \\ y^2+x\sqrt{xy}=18, \end{cases}$ to find the values of x and y.

Ans. $x=\pm 1$, and $y=\pm 4$.

9. Given $\left\{\begin{array}{l} x^2+x\sqrt[3]{xy^2}=208, \\ y^2+y\sqrt[3]{x^2y}=1053, \end{array}\right\}$ to find the values of x and y.

Ans. $x=\pm 8$, and $y=\pm 27$.

10. Given $\left\{\begin{array}{l} x^2+xy=60, \\ y^2+xy=84, \end{array}\right\}$ to find the values of x and y.

Ans. $x=\pm 5$, and $y=\pm 7$.

QUESTIONS PRODUCING PURE QUADRATICS.

QUESTION

1. There are two numbers in the proportion of 4 to 5, the difference of whose squares is 81. What are the numbers?

SOLUTION.

Let $4x$ = one of the numbers,

then $5x$ = the other number,

$$\therefore 25x^2-16x^2=81,$$
$$9x^2=81,$$
$$x^2=9,$$
$$x=\pm 3,$$

$4x=\pm 12$, one of the numbers,

$5x=\pm 15$, the other number.

QUESTION

2. What two numbers are those whose sum is to the greater as 10 to 7; and whose sum multiplied by the less produces 270?

SOLUTION.

Let $10x$ = their sum,

then $7x$ = the greater number,

and $3x$ = the less "

$$\therefore 30x^2=270,$$
$$x^2=9,$$
$$x=\pm 3,$$

$7x=\pm 21$, the greater number,

$3x=\pm 9$, the less "

QUESTIONS.

1. What two numbers are those whose difference is to the greater as 2 to 9, and the difference of whose squares is 128?

Ans. ± 18, and ± 14.

2. What three numbers are those which are in the proportion of $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$, and the sum of whose squares is 724?

Ans. ± 12, ± 16, and ± 18.

3. A merchant bought a piece of cloth for \$324; and the number of dollars he paid for a yard was to the number of yards, as 4 to 9. How many yards did he buy, and what was the price per yard?

Ans. 27 yards, at \$12 a yard.

4. A detachment from an army was marching in regular column with 5 men more in depth than in front. The front was afterwards increased by 845 men, and by this movement, the detachment was drawn up in 5 lines. How many men were in the detachment?

Ans. 4550.

5. Two partners, *A* and *B*, divided their gain, \$60, of which *B* took \$20. *A*'s money had been in trade 4 months, and if 50 be divided by the number of dollars *A* had in, the quotient will give the number of months that *B*'s money, which was \$100, had been in trade. How much money had *A* in trade, and how long had *B*'s been in trade.

Ans. *A* had \$50, and *B*'s had been in trade 1 month.

6. *A* and *B* invested some money in speculation. *A* disposes of his bargain for \$11, and gains as much per cent. as *B* invested; *B*'s gain was \$36, and the gain upon *A*'s investment was 4 times as much per cent. as upon *B*'s. How much did each invest?

Ans. *A* invested \$5, and *B* \$120.

7. A dog started in pursuit of a hare which was 7 rods ahead of him, and after running 20 rods, he observed that the hare struck off at right angles to her former course. He then changed his course so that he might overtake her without another tack. How far did the dog run, provided he ran 10 rods a minute and the hare 8, in the same time? *Ans.* 25 rods.

8. What two numbers are those whose difference multipliplied by the greater produces 40, and by the less 15? *Ans.* ± 8 and ± 3.

9. What two numbers are those whose difference multiplied by the less produces 42, and by their sum 113? 133. —

Ans. ± 13 and ± 6.

10. A man bought a field whose length was to its breadth as 8 to 5. The number of dollars that he paid for 1 acre was equal to the number of rods in the length of the field; and 13 times the number

of rods round the field equaled the number of dollars that it cost. What was the length and breadth of the field?

Ans. Length, 104 rods; breadth, 65 rods.

11. A stack of hay, whose length is to its breadth as 5 to 4, and whose height is to its breadth as 7 to 8, is worth as many cents per cubic foot as it is feet in breadth; and its whole value is 224 times as many cents as there are square feet upon the bottom. What are the dimensions of the stack?

Ans. Length, 20 feet; breadth, 16 feet; and height, 14 feet.

12. One number is m^2 times as much as another, and their product is n^2. What are the numbers?

Ans. $\pm mn$ and $\pm\frac{n}{m}$.

13. What two numbers are those which are in the ratio of 8 to 5, and whose product is 360? *Ans.* ± 24 and ± 15.

14. What two numbers are those whose sum is to their difference as 8 to 1, and the difference of whose squares is 128?

Ans. ± 18 and ± 14.

AFFECTED QUADRATICS.

(310.) AN AFFECTED QUADRATIC EQUATION is one which contains the unknown quantity in but two terms: its exponent in one being double that in the other, and the least exponent being either one, or a proper fraction whose numerator is one; as,

$$*5x^2+7x=21;\ (a-b)x^2+(c+d)x=e+f;\ x+3x^{\frac{1}{2}}=8;\ 4x^{\frac{2}{3}}+5x^{\frac{1}{3}}=13;$$
$$x^{\frac{1}{8}}+2x^{\frac{1}{16}}=3;\ x+4\sqrt{x}=7;\ \sqrt{x}+3\sqrt[4]{x}=9,\ \text{\&c.}$$

REMARK.—Various plans may be adopted for ascertaining the values of the unknown quantity in an affected quadratic equation.

In order that the most expeditious mode of solution may be adopted, special regard must be had to the peculiarities of the problem. An aptness in observing the elements of a problem which indicate its best solution can only be obtained by practice.

The student, by a careful study of the illustrative solutions which follow, will be able, it is hoped, to solve all the problems that are appended to them.

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* The student should bear in mind that $\sqrt{x}=x^{\frac{1}{2}}$; $\sqrt[3]{x}=x^{\frac{1}{3}}$; $\sqrt[4]{x}=x^{\frac{1}{4}}$, &c., and $\sqrt[3]{x^2}=x^{\frac{2}{3}}$; $\sqrt[4]{x^2}=x^{\frac{2}{4}}=x^{\frac{1}{2}}$; $\sqrt[5]{x^2}=x^{\frac{2}{5}}$, &c.

PROBLEM

1. Given $x^2-2x=-1$ (1) to find the values of x.

SOLUTION.

$$x^2-2x+1=0 \qquad (2)=(1) \text{ transposed.}$$
$$x-1=\pm 0 \qquad (3)=\sqrt{(2)}.$$

Taking the *plus* value of zero, $x-1=+0$, or $x=1$,
" " *minus* " " $x-1=-0$, or $x=1$.

This solution shows that x has two values which, in this case, are identical. This fact may be better illustrated by the following

SOLUTION.

$$x^2-2x+1=0 \qquad (2).$$
$$(x-1)(x-1)=0 \qquad (3)=(2) \text{ factored.}$$

This equation may be satisfied by placing either of the binomial factors equal to zero. Whence, we obtain the two simple equations,

$$x-1=0$$
$$\text{and } x-1=0$$

Whose solutions give $x=1$, and $x=1$.

PROBLEM

2. Given $x^2-2ax+a^2=b^2$ (1) to find the values of x.

SOLUTION.

$$x-a=\pm b \qquad (2)=\sqrt{(1)}.$$
$$x=a\pm b.$$

Whence, we find that x has two values; viz., $a+b$ and $a-b$. The same result may also be obtained by the following

SOLUTION.

Since $x^2-2ax+a^2=(x-a)^2$, Equation (1) transposed, becomes

$$(x-a)^2-b^2=0 \qquad (2)$$
$$(x-a+b)(x-a-b)=0 \qquad (3)=(2) \text{ factored.}$$

This equation may be satisfied by putting either of the above factors equal to zero. Whence, we obtain the two simple equations,

$$x-a-b=0$$
$$\text{and } x-a+b=0$$

Whose solutions give $x=a+b$ and $x=a-b$, the same as before.

EXAMPLES.

1. Given $x^2-4x+4=0$ to find x. *Ans.* $x=2$, or 2.

2. Given $x^2+2x+1=0$ to find x. *Ans.* $x=-1$, or -1.

3. Given $x^2+4x+4=0$ to find x. *Ans.* $x=-2$, or -2.

4. Given $x^2+6x+9=0$ to find x. *Ans.* $x=-3$, or -3.

5. Given $x^2-6x+9=0$ to find x. *Ans.* $x=3$, or 3.

6. Given $x^2-8x+16=0$ to find x. *Ans.* $x=4$, or 4.

7. Given $x^2+8x+16=0$ to find x. *Ans.* $x=-4$, or -4.

8. Given $x^2+10x+25=0$ to find x. *Ans.* $x=-5$, or -5.

9. Given $x^2-10x+25=0$ to find x. *Ans.* $x=5$, or 5.

10. Given $x^2-12x+36=0$ to find x. *Ans.* $x=6$, or 6.

11. Given $x^2+12x+36=0$ to find x. *Ans.* $x=-6$, or -6.

12. Given $x^2+14x+49=0$ to find x. *Ans.* $x=-7$, or -7.

13. Given $x^2-14x+49=0$ to find x. *Ans.* $x=7$, or 7

14. Given $x^2+16x+64=81$ to find x. *Ans.* $x=1$, or -17.

15. Given $x^2-18x+81=100$ to find x. *Ans.* $x=19$, or -1.

16. Given $x^2+20x+100=64$ to find x. *Ans.* $x=-2$, or -18.

17. Given $x^2=22x-121$ to find x. *Ans.* $x=11$, or 11.

18. Given $x^2-169=24x-144$ to find x. *Ans.* $x=25$, or -1.

19. Given $4x^2-4x+1=4$ to find x. *Ans.* $x=1\frac{1}{2}$, or $-\frac{1}{2}$.

20. Given $4x^2-8x+4=9$ to find x. *Ans.* $x=2\frac{1}{2}$, or $-\frac{1}{2}$.

21. Given $25x^2-20x+4=16$ to find x. *Ans.* $x=1\frac{1}{5}$, or $-\frac{2}{5}$.

22. Given $2x^2-2x+1=x^2+9$ to find x. *Ans.* $x=4$, or -2.

23. Given $4x^2-4x+4=3x^2+25$ to find x. *Ans.* $x=7$, or -3.

24. Given $6x^2-6x+9=5x^2+1$ to find x. *Ans.* $x=4$, or 2.

25. Given $9x^2-8x+16=8x^2+25$ to find x. *Ans.* $x=9$, or -1.

26. Given $5x^2-12x+9=x^2+36$ to find x. *Ans.* $x=4\frac{1}{2}$, or $-1\frac{1}{2}$.

27. Given $100x^2-36x+4=19x^2+49$ to find the value of x.
Ans. $x=1$, or $-\frac{5}{9}$.

28. Given $x^2-4x+1=-2x+9$ to find x. *Ans.* $x=4$, or -2.

29. Given $4x^2+9x+16=4-7x$ to find x. *Ans.* $x=-1$, or -3.

30. Given $5x^2-6x+4=4x^2-2x+16$ to find the value of x.
Ans. $x=6$, or -2.

31. Given $9x^2+12x-25=5x^2+4x-4$ to find the value of x.
Ans. $x=1\frac{1}{2}$, or $-3\frac{1}{2}$.

32. Given $25x^2+20x-81=9x^2+4x-4$ to find the value of x.
Ans. $x=1\frac{3}{4}$, or $-2\frac{3}{4}$.

33. Given $x^2-2ax+a^2=b$ to find x. *Ans.* $x=a\pm\sqrt{b}$.

34. Given $\frac{1}{9}x^2+\frac{1}{3}x-9=-\frac{1}{4}$ to find x. *Ans.* $x=7\frac{1}{2}$, or $-10\frac{1}{2}$.

35. Given $\dfrac{4}{x^2-2x+1}=\frac{1}{4}$ to find x. *Ans.* $x=5$, or -3.

36. Given $\frac{1}{64}x^2+x+16=16x^2$ to find x. *Ans.* $x=1\frac{1}{31}$, or $-\frac{32}{33}$.

37. Given $\dfrac{x^2}{9}+\dfrac{x}{3}+\dfrac{1}{4}=12$ to find x. *Ans.* $x=-\frac{3}{2}\pm6\sqrt{3}$.

38. Given $\dfrac{a^2x^2}{f^2}-\dfrac{2ax}{g}+\dfrac{f^2}{g^2}=0$ to find x. *Ans.* $x=\dfrac{f^2}{ag}$, or $\dfrac{f^2}{ag}$.

39. Given $cx^2-2cx\sqrt{d}=dx^2-cd$ to find x. *Ans.* $x=\dfrac{\sqrt{cd}}{\sqrt{c}\mp\sqrt{d}}$.

40. Given $mx^2+mn=2mx\sqrt{n}+nx^2$ to find x. *Ans.* $x=\dfrac{\sqrt{mn}}{\sqrt{m}\mp\sqrt{n}}$.

41. Given $abx^2-2x(a+b)\sqrt{ab}=(a-b)^2$ to find the value of x.
Ans. $x=\dfrac{a+b\pm\sqrt{2a^2+2b^2}}{\sqrt{ab}}$.

42. Given $ax^2+b^2+c^2=a^2+2bc+2(b-c)x\sqrt{a}$ to find the value of x.
Ans. $x=\dfrac{b-c\pm a}{\sqrt{a}}$.

43. Given $x^2-x=-\frac{1}{4}$ to find the value of x. *Ans.* $x=\frac{1}{2}$, or $\frac{1}{2}$.

44. Given $x^2-10x=-25$ to find x. *Ans.* $x=5$, or 5.

(311.) In order to solve the following problems, the student should be familiar with the modes of eliminating radicals, and should also be able at sight to detect the squares of binomials.

PROBLEM

1. Given $\sqrt{\dfrac{x+a}{x}}+2\sqrt{\dfrac{a}{x+a}}=b^2\sqrt{\dfrac{x}{x+a}}$ (1) to find the values of x.

SOLUTION.

$$\frac{x+a}{x}+2\sqrt{\frac{a}{x}}=b^2 \qquad (2)=(1)\times\sqrt{\frac{x+a}{x}}$$

The student might not be able to see that the left hand member of this equation is a perfect square, and not observing this, the solution would be much more difficult for him. Let us endeavor to render this fact more apparent.

$$1+\frac{a}{x}+2\sqrt{\frac{a}{x}}=b^2 \qquad (3)=(2) \text{ divided.}$$

$$1+2\sqrt{\frac{a}{x}}+\frac{a}{x}=b^2 \qquad (4)=(3) \text{ arranged.}$$

$$1+\sqrt{\frac{a}{x}}=\pm b \qquad (5)=\sqrt{(4)}.$$

$$1\mp b=-\sqrt{\frac{a}{x}}.$$

$$(1\mp b)^2=\frac{a}{x}.$$

$$x=\frac{a}{(1\mp b)^2}.$$

PROBLEM

2. Given $\frac{3x-1}{\sqrt{3x}+1}=1+\frac{\sqrt{3x}-1}{2}$ to find the values of x.

Ans. $x=3$, or $\frac{1}{3}$.

SOLUTION.

$$\text{Since, } 3x-1=(\sqrt{3x}+1)(\sqrt{3x}-1),$$

$$\frac{3x-1}{\sqrt{3x}+1}=\sqrt{3x}-1,$$

$$\text{whence, } \sqrt{3x}-1=1+\frac{\sqrt{3x}-1}{2},$$

$$2\sqrt{3x}=4+\sqrt{3x}-1,$$

$$\sqrt{3x}=3,$$

$$3x=9,$$

$$x=3.$$

But how is the value $\frac{1}{3}$ obtained? By putting $\sqrt{3x}+1$, which is used in reducing the fraction in the first member of the equation, equal to zero.

Thus,
$$\sqrt{3x}+1=0$$
$$\sqrt{3x}=-1,$$
$$3x=1,$$
$$x=\tfrac{1}{3}.$$

But if we attempt to verify the given equations by substituting $\frac{1}{3}$ for the value of x, we obtain $\frac{0}{2}=1+\frac{0}{2}$.

Is this a true equation? It seems not; for by dropping $\frac{0}{2}$ from both members, we obtain $0=1$, an absurdity.

Are we then to conclude that $x=\frac{1}{3}$ is not true?

In substituting $\frac{1}{3}$ for x, we assumed that $\sqrt{3x}=+1$, whereas, we learn from the equation

$$\sqrt{3x}+1=0$$
$$\text{that } \sqrt{3x}=-1.$$

If we substitute, keeping in mind the fact that $\sqrt{3x}$ must be taken equal to -1, we shall obtain

$$\frac{0}{0}=1+\frac{-2}{2}=1-1$$
$$\frac{0}{0}=0, \text{ or } 0=0,$$

from which we learn that $x=\frac{1}{3}$ is a true value of x.

From this solution we may infer the following

PRINCIPLE.

Any expression containing the unknown quanity that will divide both the numerator and the denominator of any fraction in the equation, being placed equal to zero, will give as many values of the unknown quantity of the given equation as the unknown quantity in the equation thus formed has values.

Since, a factor of the kind referred to in this principle may by multiplication become common to both members of the equation, we obtain the following

PRINCIPLE.

Any expression containing the unknown quantity that will divide both members of an equation, being placed equal to zero, will form an equation of which every value of the unknown quantity will also be a value of the unknown quantity in the given equation.

EXAMPLES.

1. Given $\frac{5x-9}{\sqrt{5x}+3}=1+\frac{\sqrt{5x}-3}{2}$ to find the values of x.

Ans. $x=5$, or $\frac{9}{5}$.

2. Given $\frac{ax-b^2}{\sqrt{ax}+b}=c+\frac{\sqrt{ax}-b}{c}$ to find the values of x.

Ans. $x=\frac{1}{a}\left(b+\frac{c^2}{c-1}\right)^2$, or $\frac{b^2}{a}$

3. Given $\sqrt[3]{64+x^2-8x}=\frac{4+x}{\sqrt[3]{4+x}}$ to find the values of x.

Ans. $x=3$, or -4.

4. Given $\sqrt[3]{a^2+x^2-2ax}=\frac{a+x}{\sqrt[3]{a+x}}$ to find the values of x.

Ans. $x=\frac{a(a-1)}{4}$, or $-a$.

5. Given $\frac{a+x+\sqrt{2ax+x^2}}{a+x}=b$ to find the values of x.

Ans. $x=-a\pm\frac{a}{2b-b^2}\sqrt{2b-b^2}$.

PROBLEM

(312.) 1. Given $\left\{\begin{array}{r} x+y=s \;(1), \\ xy=a^2 \;(2), \end{array}\right\}$ to find the values of x and y.

SOLUTION.

$$\begin{array}{rl}
x^2+2xy+y^2=s^2 & (3)=(1)^2. \\
4xy=4a^2 & (4)=(2)\times 4. \\
x^2-2xy+y^2=s^2-4a^2 & (5)=(3)-(4). \\
x-y=\pm\sqrt{s^2-4a^2} & (6)=\sqrt{5}. \\
2x=s\pm\sqrt{s^2-4a^2} & (7)=(6)+(1). \\
2y=s\mp\sqrt{s^2-4a^2} & (8)=(1)-(6). \\
x=\frac{1}{2}(s\pm\sqrt{s^2-4a^2}). & \\
y=\frac{1}{2}(s\mp\sqrt{s^2-4a^2}). &
\end{array}$$

PROBLEM

2. Given $\left\{\begin{array}{r} x-y=1 \;(1), \\ x^3-y^3=7 \;(2), \end{array}\right\}$ to find the values of x and y.

SOLUTION.

$$x^2+xy+y^2=7 \qquad (3)=(2)\div(1).$$
$$x^2-2xy+y^2=1 \qquad (4)=(1)^2.$$
$$3xy=6 \qquad (5)=(3)-(4).$$
$$xy=2 \qquad (6)=(5)\div 3.$$
$$x^2+2xy+y^2=9 \qquad (7)=(3)+(6).$$
$$x+y=\pm 3 \qquad (8)=\sqrt{(7)}.$$
$$2x=4, \text{ or } -2 \qquad (9)=(8)+(1).$$
$$2y=2, \text{ or } -4 \qquad (10)=(8)-(1).$$
$$x=2, \text{ or } -1 \qquad (11)=(9)\div 2.$$
$$y=1, \text{ or } -2 \qquad (12)=(10)\div 2.$$

REMARK.—Different artifices may be employed in eliminating one of the unknown quantities in the following equations. The student should, however, be careful not to find the value of one of the unknown quantities in one equation, and substitute it in the other, for an equation would then arise which he is not yet prepared to solve.

EXAMPLES.

1. Given $\left\{\begin{array}{r} x+y=3 \\ xy=2 \end{array}\right\}$ to find the values of x, and y.

Ans. $x=2$, or 1; $y=1$, or 2.

2. Given $\left\{\begin{array}{r} x+y=8 \\ xy=15 \end{array}\right\}$ to find the values of x, and y.

Ans. $x=5$, or 3; $y=3$, or 5.

3. Given $\left\{\begin{array}{r} x+y=33 \\ xy=266 \end{array}\right\}$ to find the values of x and y.

Ans. $x=19$, or 14; $y=14$, or 19.

4. Given $\left\{\begin{array}{r} x-y=2 \\ x^3-y^3=26 \end{array}\right\}$ to find the values of x and y.

Ans. $x=3$, or -1; $y=1$, or -3.

5. Given $\left\{\begin{array}{r} x-y=a \\ x^3-y^3=b \end{array}\right\}$ to find the values of x and y.

$$Ans. \left\{\begin{array}{l} x=\frac{a}{2}\pm\frac{1}{2}\sqrt{\frac{3b+b-a^3}{3a}}. \\ y=\frac{a}{2}\mp\frac{1}{2}\sqrt{\frac{3b+b-a^3}{3a}}. \end{array}\right.$$

6. Given $\left\{\begin{array}{r} x+y=3 \\ x^2+y^2=5 \end{array}\right\}$ to find the values of x and y.

Ans. $x=2$, or 1; $y=1$, or 2.

7. Given $\left\{\begin{matrix} x+y=s \\ x^2+y^2=a^2 \end{matrix}\right\}$ to find the values of x and y.

Ans. $\left\{\begin{matrix} x=\frac{1}{2}s\pm\frac{1}{2}\sqrt{2a^2-s^2}, \\ y=\frac{1}{2}s\mp\frac{1}{2}\sqrt{2a^2-s^2}. \end{matrix}\right.$

8. Given $\left\{\begin{matrix} x^{\frac{1}{3}}+y^{\frac{1}{3}}=5 \\ x^{\frac{2}{3}}+y^{\frac{2}{3}}=13 \end{matrix}\right\}$ to find the values of x and y.

Ans. $x=27$, or 8; $y=8$, or 27.

9. Given $\left\{\begin{matrix} x-y=1 \\ x^2+y^2=5 \end{matrix}\right\}$ to find the values of x and y.

Ans. $x=2$, or -1; $y=1$, or -2.

10. Given $\left\{\begin{matrix} x-y=a \\ x^2+y^2=b \end{matrix}\right\}$ to find the values of x and y.

Ans. $\left\{\begin{matrix} x=\frac{1}{2}a\pm\frac{1}{2}\sqrt{2b-a^2} \\ y=-\frac{1}{2}a\pm\frac{1}{2}\sqrt{2b-a^2}. \end{matrix}\right.$

11. Given $\left\{\begin{matrix} x+y=4 \\ x^3+y^3=(x+y)^2 \end{matrix}\right\}$ to find the values of x and y.

Ans. $x=2$, or 2; $y=2$, or 2.

12. Given $\left\{\begin{matrix} x^{\frac{1}{2}}+y^{\frac{1}{2}}=3 \\ x^{\frac{1}{2}}y^{\frac{1}{2}}=2 \end{matrix}\right\}$ to find the values of x and y.

Ans. $x=4$, or 1; $y=1$, or 4.

13. Given $\left\{\begin{matrix} x^{\frac{1}{2}}-y^{\frac{1}{2}}=2 \\ x^{\frac{3}{2}}-y^{\frac{3}{2}}=26 \end{matrix}\right\}$ to find the values of x and y.

Ans. $x=9$, or 1; $y=1$, or 9.

14. Given $\left\{\begin{matrix} x^{\frac{1}{2}}+y^{\frac{1}{2}}=3 \\ x+y=5 \end{matrix}\right\}$ to find the values of x and y.

Ans. $x=4$, or 1; $y=1$, or 4.

15. Given $\left\{\begin{matrix} x^{\frac{1}{2}}-y^{\frac{1}{2}}=1 \\ x+y=5 \end{matrix}\right\}$ to find the values of x and y.

Ans. $x=4$, or 1; $y=1$, or 4.

16. Given $\left\{\begin{matrix} x^{\frac{1}{2}}+y^{\frac{1}{2}}=4 \\ x^{\frac{3}{2}}+y^{\frac{3}{2}}=(x^{\frac{1}{2}}+y^{\frac{1}{2}})^2 \end{matrix}\right\}$ to find the values of x and y.

Ans. $x=4$, or 4; $y=4$, or 4.

17. Given $\left\{\begin{matrix} x^{\frac{1}{8}}+y^{\frac{1}{8}}=3 \\ x^{\frac{1}{8}}y^{\frac{1}{8}}=2 \end{matrix}\right\}$ to find the values of x and y.

Ans. $x=256$, or 1; $y=1$, or 256.

18. Given $\left\{\begin{array}{l} x^{\frac{1}{8}}-y^{\frac{1}{8}}=2 \\ x^{\frac{3}{8}}-y^{\frac{3}{8}}=26 \end{array}\right\}$ to find the values of x and y.

Ans. $x=6561$, or 1; $y=1$, or 6561.

19. Given $\left\{\begin{array}{l} x^{\frac{1}{8}}-y^{\frac{1}{8}}=1 \\ x^{\frac{1}{4}}+y^{\frac{1}{4}}=5 \end{array}\right\}$ to find the values of x and y.

Ans. $x=256$, or 1; $y=1$, or 256.

(313.) Every affected quadratic equation can be reduced to the following form:

$$x^2 \pm Ax = \pm B,$$

In which A and B may represent any quantities whatever, whether real or imaginary, whole or fractional.

PROBLEM.

To solve $x^2 \pm Ax = \pm B$.

SOLUTION.

First, let us examine the case in which A is an even whole number. Putting $A=2a$, and $B=b$, and using the *plus* signs, since the principle upon which the solution depends is the same as when A and B are both *minus*, or either one plus and the other minus, we have

$$x^2+2ax=b. \qquad (1)$$

We see by the principles of binomial squares that the first member of (1) would be a perfect square if it were increased by a^2. Let us then add a^2 to the first member, and, to preserve the equality, we must also add it to the second member. We then have

$$x^2+2ax+a^2=a^2+b \qquad (2)$$

Extracting the square root, $x+a=\pm\sqrt{a^2+b}$

$$x=-a\pm\sqrt{a^2+b}$$

This problem may also be solved in the following manner:

By transposition (2) becomes $(x+a)^2-(a^2+b)=0 \qquad (3)$

Considering a^2+b as the square of $\sqrt{a^2+b}$, the left hand member of (3) is the difference of two squares, and consequently may be factored thus:

$$(x+a-\sqrt{a^2+b})(x+a+\sqrt{a^2+b})=0$$

This equation may be satisfied by putting either factor equal to zero, whence result the two simple equations:

$$x+a-\sqrt{a^2+b}=0$$

$$\text{and } x+a+\sqrt{a^2+b}=0$$

The first of these simple equations gives

$$x=-a+\sqrt{a^2+b}$$

And the second, $x=-a-\sqrt{a^2+b}$, the same as before.

EXAMPLES.

1. Given $x^2+8x=48$ to find x. *Ans.* $x=4$, or -12.

2. Given $x^2+4x=140$ to find x. *Ans.* $x=10$, or -14.

3. Given $x^2-6x=-8$ to find x. *Ans.* $x=4$, or $+2$.

4. Given $x^2+8x=33$ to find x. *Ans.* $x=3$, or -11.

5. Given $x^2-10x=-21$ to find x. *Ans.* $x=7$, or 3.

6. Given $x^2+8x=65$ to find x. *Ans.* $x=5$, or -13.

7. Given $x^2-2px=q$ to find x. *Ans.* $x=p\pm\sqrt{p^2+q}$.

8. Given $x^2+12x=108$ to find x. *Ans.* $x=6$, or -18.

9. Given $x^2-14x=51$ to find x. *Ans.* $x=17$, or -3.

10. Given $x^2-8x=48$ to find x. *Ans.* $x=12$, or -4.

11. Given $x^2+10x=-24$ to find x. *Ans.* $x=-4$, or -6.

12. Given $x^2+16x=-55$ to find x. *Ans.* $x=-5$, or -11.

(314.) Equations of the form $x^{\frac{2}{n}}\pm 2ax^{\frac{1}{n}}=\pm b$, n being integral, are affected quadratics, and should be solved as the preceding examples.

In such equations, if we consider, primarily, that $x^{\frac{1}{n}}$ is the unknown quantity, as we should do, the above equation becomes of the general form, which we have already discussed; for, putting $y=x^{\frac{1}{n}}$, $y^2=x^{\frac{2}{n}}$, and substituting these values, we get the equation,

$$y^2\pm 2ay=\pm b$$

which is the general form referred to.

The student should observe that, in some of the following examples, $x^{\frac{1}{n}}=\pm\sqrt[n]{x}$, when n is an even number, should be taken with the *minus* sign, in order to verify the equation.

PROBLEM

1. Given $x+8x^{\frac{1}{2}}=48$ (1) to find the values of x.

SOLUTION.

$x+8x^{\frac{1}{2}}+16=64$	(2)=(1) with 16 added to both members.
$x^{\frac{1}{2}}+4=\pm 8$	$(3)=\sqrt{(2)}$.
$x^{\frac{1}{2}}=4$, or -12	(4)=(3) transposed.
$x=16$, or 144	$(5)=(4)^2$.

The equations (4) and (5) show, that in attempting to verify the original equation with 16, that $x^{\frac{1}{2}}$ must be taken equal to 4; but, when attempting to verify it with 144, $x^{\frac{1}{2}}$ must be taken equal to -12, and not $+12$.

Let us solve another

PROBLEM

2. Given $x-6x^{\frac{1}{2}}=-8$ (1) to find the values of x.

SOLUTION.

$x-6x^{\frac{1}{2}}+9=1$	(2)=(1) with 9 added to both members.
$x^{\frac{1}{2}}-3=\pm 1$	$(3)=\sqrt{(2)}$.
$x^{\frac{1}{2}}=4$, or 2	(4)=(3) transposed.
$x=16$, or 4	$(5)=(4)^2$.

In this example, in verifying the values of x; for $x=16$, $x^{\frac{1}{2}}$ must be taken equal to 4; and for $x=4$, $x^{\frac{1}{2}}$ must be taken equal to 2, and this is just what the student would be likely to do.

We learn from these solutions that, in verifying equations, the values of $x^{\frac{1}{n}}$, (n being even, as in the above problems,) must be carefully observed.

EXAMPLES.

1. Given $x+2x^{\frac{1}{2}}=8$ to find $x^{\frac{1}{2}}$. *Ans.* $x^{\frac{1}{2}}=2$, or -4.

2. Given $x+6x^{\frac{1}{2}}=16$ to find $x^{\frac{1}{2}}$. *Ans.* $x^{\frac{1}{2}}=2$, or -8.

3. Given $x+4x^{\frac{1}{2}}=12$ to find $x^{\frac{1}{2}}$. *Ans.* $x^{\frac{1}{2}}=2$, or -6.

4. Given $x-8x^{\frac{1}{2}}=48$ to find $x^{\frac{1}{2}}$. *Ans.* $x^{\frac{1}{2}}=12$, or -4.

5. Given $x+10x^{\frac{1}{2}}=3$ to find $x^{\frac{1}{2}}$. *Ans.* $x^{\frac{1}{2}}=-5\pm 2\sqrt{7}$.

6. Given $x^{\frac{2}{3}}+12x^{\frac{1}{3}}=13$ to find $x^{\frac{1}{3}}$. *Ans.* $x^{\frac{1}{3}}=1$, or -13.

7. Given $x^{\frac{1}{2}}+14x^{\frac{1}{4}}=15$ to find $x^{\frac{1}{4}}$. *Ans.* $x^{\frac{1}{4}}=1$, or -15.

8. Given $x^{\frac{2}{5}}+16x^{\frac{1}{5}}=17$ to find $x^{\frac{1}{5}}$. *Ans.* $x^{\frac{1}{5}}=1$, or -17.

9. Given $x^{\frac{1}{50}}+18x^{\frac{1}{100}}=19$ to find the values of $x^{\frac{1}{100}}$.

Ans. $x^{\frac{1}{100}}=1$, or -19.

10. Given $x+20\sqrt{x}=21$ to find $\sqrt{x}$. *Ans.* $\sqrt{x}=1$, or -21.

11. Given $\sqrt[3]{x^2}+22\sqrt[3]{x}=23$ to find $\sqrt[3]{x}$. *Ans.* $\sqrt[3]{x}=1$, or -23.

12. Given $\sqrt{x}+24\sqrt[4]{x}=25$ to find $\sqrt[4]{x}$. *Ans.* $\sqrt[4]{x}=1$, or -25.

(315.) The solution of these examples may be somewhat abridged by omitting the formality of completing the square.

PROBLEM.

Given $x^2-6x+19=13$ (1) to find the values of x.

SOLUTION.

$x^2-6x=-6$	(2)=(1) transposed.
$x^2-6x+9=3$	(3)=(2) with 9 added to both members.
$x-3=\pm\sqrt{3}$	(4)=$\sqrt{(3)}$.
$x=3\pm\sqrt{3}$	(5)=(4) transposed.

Equation (5) may be written immediately from (2); since x is found to be equal to *half the coefficient of x taken with a contrary sign,* PLUS *or* MINUS *the square root of the known term after it has been increased by the square of half the coefficient of x.* To prove this fact to be general, let us assume the four general equations:

$$x^2+2ax=b$$
$$x^2-2ax=b$$
$$x^2+2ax=-b$$
$$x^2-2ax=-b$$

A solution of these four equations gives the following values of x, respectively:

$$x=-a\pm\sqrt{a^2+b}$$
$$x=\quad a\pm\sqrt{a^2+b}$$
$$x=-a\pm\sqrt{a^2-b}$$
$$x=\quad a\pm\sqrt{a^2-b}$$

From an examination of these four equations and the values of the unknown quantity in each, we derive the following

RULE.

When an affected quadratic equation is reduced to the form

$x^2 \pm 2ax = \pm b$, the value of the unknown quantity may be found by putting it equal to half the coefficient of its first power taken with a contrary sign, plus or minus the square root of the unknown term after it has been added to the square of half the coefficient of the first power of the unknown quantity.

REMARK.—The student should apply this rule in the solution of the following examples after they are reduced to the proper form.

EXAMPLES.

1. Given $x^2-6x+19=11$ to find x. *Ans.* $x=2$, or 4.

2. Given $x^2+6bx=c^2$ to find x. *Ans.* $x=-3b\pm\sqrt{9b^2+c^2}$.

3. Given $\frac{x}{a}+\frac{a}{x}=\frac{2}{a}$ to find the values of x. *Ans.* $x=1\pm\sqrt{1-a^2}$.

4. Given $\frac{3x}{2}+\frac{2}{3x}=x+\frac{2x-2}{3}$ to find x. *Ans.* $x=2\pm2\sqrt{2}$.

5. Given $x^2+12x-16=92$ to find x. *Ans.* $x=6$, or -18.

6. Given $x^2+6x+4=59$ to find x. *Ans.* $x=5$, or -11.

7. Given $x^2-8x+10=19$ to find x. *Ans.* $x=9$, or -1.

8. Given $x^2-12x+30=3$ to find x. *Ans.* $x=9$, or 3.

9. Given $x^2+6x=27$ to find x. *Ans.* $x=3$, or -9.

10. Given $8x^2+32x=360$ to find x. *Ans.* $x=5$, or -9.

11. Given $x^2-8x=14$ to find x. *Ans.* $x=4\pm\sqrt{30}$.

12. Given $2x^2+8x-20=70$ to find x. *Ans.* $x=5$, or -9.

13. Given $4x^2-8x+6=326$ to find x. *Ans.* $x=10$, or -8.

14. Given $x+6x^{\frac{1}{2}}=27$ to find x. *Ans.* $x=9$, or 81.

15. Given $\sqrt{x}-4\sqrt[4]{x}=9$ to find x. *Ans.* $x=497\pm136\sqrt{13}$.

16. Given $x-8\sqrt{x}=14$ to find x. *Ans.* $x=46\pm8\sqrt{30}$.

(316.) Every affected quadratic equation may be reduced to the form $cx^2\pm2ax=\pm b$, in which c, a, and b are whole numbers or surds.

The simplest case of this general form, which is when $c=1$, has already been treated of.

PROBLEM

1. Given $cx^2+2ax=b$ (1) to find the values of x.

SOLUTION.

$c^2x^2+2acx=bc$ $\quad(2)=(1)\times c.$

$c^2x^2+2acx+a^2=a^2+bc$ $\quad(3)=(2)$ with a^2 added to both members.

$cx+a=\pm\sqrt{a^2+bc}$ $\quad(4)=\sqrt{(3)}.$

$cx=-a\pm\sqrt{a^2+bc}$ $\quad(5)=(4)$ transposed.

$x=\dfrac{-a\pm\sqrt{a^2+bc}}{c}$ $\quad(6)=(5)\div c.$

PROBLEM

2. Given $x^2-3x=40$ (1) to find the values of x.

SOLUTION.

As the coefficient of x is odd, this equation is not of the requsite form, but in all such cases it may be made so by multiplying by 2.

$2x^2-6x=80$ $\quad(2)=(1)\times 2.$

$4x^2-12x=160$ $\quad(3)=(2)\times 2.$

$4x^2-12x+9=169$ $\quad(4)=(3)$ with square completed.

$2x-3=\pm 13$ $\quad(5)=\sqrt{(4)}.$

$2x=16$, or -10 $\quad(6)=(5)$ transposed.

$x=8$, or -5 $\quad(7)=(6)\div 2.$

EXAMPLES.

1. Given $3x^2+2x=85$ to find x. *Ans.* $x=5$, or $-5\frac{2}{3}$.

2. Given $3x^2+4x=340$ to find x. *Ans.* $x=10$, or $-11\frac{1}{3}$.

3. Given $5x^2+6x=63$ to find x. *Ans.* $x=3$, or $-4\frac{1}{5}$.

4. Given $3x^2-14x=-15$ to find x. *Ans.* $x=3$, or $1\frac{2}{3}$.

5. Given $4x^2-6x=108$ to find x. *Ans.* $x=6$, or $-4\frac{1}{2}$.

6. Given $3x^2-2x=65$ to find x. *Ans.* $x=5$, or $-4\frac{1}{3}$.

7. Given $15x^2-622x=-6384$ to find x. *Ans.* $x=22\frac{4}{5}$, or $18\frac{2}{3}$.

8. Given $\dfrac{x^2}{3}+\dfrac{4x}{5}-19=15\frac{1}{5}$ to find x. *Ans.* $x=9$, or $-11\frac{2}{5}$.

9. Given $118x-2\frac{1}{2}x^2=20$ to find x. *Ans.* $x=\dfrac{118\pm\sqrt{13724}}{5}$.

10. Given $7x^2-20x=32$ to find x. *Ans.* $x=4$, or $-1\frac{1}{7}$.

11. Given $5x^2+4x=273$ to find x. *Ans.* $x=7$, or $-7\frac{4}{5}$.

12. Given $abx^2-2x(a+b)\sqrt{ab}=(a-b)^2$ to find x.

Ans. $\dfrac{a+b\pm\sqrt{2a^2+2b^2}}{\sqrt{ab}}$,

(317.) The student, after solving the examples in the last article, is presumed to be fully acquainted with the principles of their solution, and is, therefore, prepared to omit some of the intermediate equations. The general form,

$$cx^2 \pm 2ax = \pm b,$$

may be divided into the four following equations:

$$cx^2 + 2ax = b,$$
$$cx^2 - 2ax = b,$$
$$cx^2 + 2ax = -b,$$
$$cx^2 - 2ax = -b,$$

whose solutions give, respectively, the following values for x:

$$x = \frac{-a \pm \sqrt{a^2 + bc}}{c},$$

$$x = \frac{a \pm \sqrt{a^2 + bc}}{c},$$

$$x = \frac{-a \pm \sqrt{a^2 - bc}}{c},$$

$$x = \frac{a \pm \sqrt{a^2 - bc}}{c}.$$

A comparison of these values with the equations from which they are derived, gives the following

RULE.

When an affected quadratic equation is reduced to one of the four forms indicated by the general equation, $cx^2 \pm 2ax = \pm b$, the values of the unknown quantity may be found by putting it equal to half the coefficient of x, taken with a contrary sign, PLUS *or* MINUS *the square root of the product of the known term by the coefficient of x^2, after this product has been increased by the square of half the coefficient of x, and then dividing the whole by the coefficient of x^2.*

REMARK.—In the following examples, when reduced to the proper form, the student should write immediately the values of x, being guided by the rule just given.

PROBLEM.

Given $3x^2 - 30 = 4x + 2$ to find the values of x.

SOLUTION.

$$3x^2 - 4x = 32$$

$$x = \frac{2 \pm 10}{3} = 4, \text{ or } -2\tfrac{2}{3}.$$

EXAMPLES.

1. Given $x^2-7x=-3\frac{1}{4}$ to find the values of x. *Ans.* $x=6\frac{1}{2}$, or $\frac{1}{2}$.

2. Given $2x^2-10x+7=-5$ to find x. *Ans.* $x=3$, or 2.

3. Given $3x^2+4x-7=88$ to find x. *Ans.* $x=5$, or $-6\frac{1}{3}$.

4. Given $11x^2-100x=-201$ to find x. *Ans.* $x=3$, or $6\frac{1}{11}$.

5. Given $\frac{x^2}{5}+20x=3x^2-80$ to find x. *Ans.* $x=10$ or $-2\frac{6}{7}$.

6. Given $\frac{x}{5}+\frac{5}{x}=5\frac{1}{5}$ to find x. *Ans.* $x=25$, or 1.

7. Given $21x^2-1616x=-20748$ to find x. *Ans.* $x=60\frac{2}{3}$, or $16\frac{2}{7}$.

8. Given $\frac{18x^2}{5}+\frac{18078x}{65}=-4728$ to find x.

Ans. $x=-25\frac{10}{39}$, or -52.

9. Given $4x^2-9x=5x^2-255\frac{3}{4}-8x$ to find the values of x.

Ans. $15\frac{1}{2}$, or $-16\frac{1}{2}$.

10. Given $(4a^2-9cd^2)x^2+(4a^2c^2+4abd^2)x=-(ac^2+bd^2)^2$ to find the values of x. *Ans.* $x=-\frac{ac^2+bd^2}{2a\pm 3d\sqrt{c}}$.

11. Given $6x^2+2x=14$ to find x. *Ans.* $x=\frac{-1\pm\sqrt{85}}{6}$.

12. Given $32a^{2m}c^{n-1}+4a^{m+3}c^{n-1}(ac^3-2)x=a^7c^{n+2}x^2$ to find the values of x. *Ans.* $x=4a^{m-3}$, or $-\frac{8a^{m-4}}{c^3}$.

(318.) Let us now examine the general equation $cx^2\pm ax=\pm b$, in which a is an odd number. It is evident that the rule given in the last article is also applicable here, but in order to avoid multiplying the equation by 2, or in case this should not be done, to avoid fractions, we seek another mode of solution.

PROBLEM.

Given $cx^2+ax=b$ to find the values of x.

SOLUTION.

If we multiply this equation by $4c$, we shall have

$$4c^2x^2+4acx=4bc,$$

$$4c^2x^2+4acx+a^2=a^2+4bc,$$

$$2cx+a=\pm\sqrt{a^2+4bc},$$

$$2cx=-a\pm\sqrt{a^2+4bc},$$

$$x=\frac{-a\pm\sqrt{a^2+4bc}}{2c}.$$

This mode of solution is found in the *Bija Ganita*, a Hindoo Treatise on Algebra, which has been translated by Mr. Colebrooke.

PROBLEM.

Given $2x^2-5x=117$ (1) to find the values of x.

SOLUTION.

$$16x^2-40x=936 \qquad (2)=(1)\times 8.$$
$$16x^2-40x+25=961.$$
$$4x-5=\pm 31.$$
$$4x=5\pm 31=36, \text{ or } -26.$$
$$x=9, \text{ or } -6\tfrac{1}{2}.$$

EXAMPLES.

1. Given $x^2-34=\frac{1}{3}x$ to find x. *Ans.* $x=6$, or $-5\frac{2}{3}$.

2. Given $x^2+3x=72$ to find x. *Ans.* $x=\dfrac{-3\pm 3\sqrt{32}}{2}$.

3. Given $5x^2+x=4$ to find x. *Ans.* $x=\frac{4}{5}$, or -1.

4. Given $2x^2-x=21$ to find x. *Ans.* $x=3\frac{1}{2}$, or -3.

5. Given $ax^2-bx=c$ to find x. *Ans.* $x=\dfrac{b\pm\sqrt{b^2+4ac}}{2a}$.

6. Given $x^2+px=q$ to find x. *Ans.* $x=\dfrac{-p\pm\sqrt{p^2+4q}}{2}$.

7. Given $3x^2+5x=42$ to find x. *Ans.* $x=3$, or $-4\frac{2}{3}$.

8. Given $x^2+6x+4=22-x$ to find x. *Ans.* $x=2$, or -9.

9. Given $x^2-5\frac{3}{4}x=18$ to find x. *Ans.* $x=8$, or $-2\frac{1}{4}$.

10. Given $x^2-3x=10$ to find x. *Ans.* $x=5$, or -2.

11. Given $5x^2-\dfrac{x}{2}=78$ to find x. *Ans.* $x=4$, or $-3\frac{9}{10}$.

12. Given $4x^{\frac{1}{3}}+x^{\frac{1}{6}}=39$ to find x. *Ans.* $x=729$, or $\left(\dfrac{13}{4}\right)^6$.

(319.) This mode of solution may be abridged by omitting the intermediate equations. The relation of the unknown quantity to the known terms of the general equation $cx^2\pm ax=\pm b$, may be observed by solving the following equations:

$$cx^2+ax=b,$$
$$cx^2-ax=b,$$
$$cx^2+ax=-b,$$
$$cx^2-ax=-b.$$

The values of x in these four equations are

$$x=\frac{-a\pm\sqrt{a^2+4bc}}{2c},$$

$$x=\frac{a\pm\sqrt{a^2+4bc}}{2c},$$

$$x=\frac{-a\pm\sqrt{a^2-4bc}}{2c},$$

$$x=\frac{a\pm\sqrt{a^2-4bc}}{2c}.$$

By a comparison of these values with the equations from which they are derived, we obtain the following

RULE.

When an affected quadratic equation is reduced to one of the forms indicated by the general equation

$$cx^2\pm ax=\pm b,$$

the value of the unknown quantity may be found by putting it equal to the coefficient of x, taken with a contrary sign plus or minus the square root of the square of the coefficient of x, after it has been added to four times the product of the coefficient of x^2 by the known term, and dividing the whole by twice the coefficient of x^2.

PROBLEM.

Given $8x^2-7x=-34$ to find the values of x.

SOLUTION.

$$x=\frac{7\pm\sqrt{-1039}}{16}.$$

EXAMPLES.

1. Given $6x^2-x=92$ to find x. *Ans.* $x=4$, or $-3\frac{5}{6}$.

2. Given $8x^2-7x=165$ to find x. *Ans.* $x=5$, or $-4\frac{1}{8}$.

3. Given $3x^2-3x+6=5\frac{1}{3}$ to find x. *Ans.* $x=\frac{1}{3}$, or $\frac{2}{3}$.

4. Given $11\frac{3}{4}x-3\frac{1}{2}x^2=-41\frac{1}{4}$ to find x. *Ans.* $x=-2\frac{1}{7}$, or $5\frac{1}{2}$.

5. Given $9\frac{1}{3}x^2-90\frac{1}{3}x=-195$ to find x. *Ans.* $x=6\frac{3}{7}$, or $3\frac{1}{4}$.

6. Given $adx-acx^2=bcx-bd$ to find x. *Ans.* $x=\frac{d}{c}$, or $-\frac{b}{a}$.

7. Given $(a+b)x^2=cx+\frac{ac}{a+b}$ to find x. *Ans.* $x=\frac{c\pm\sqrt{c^2+4ac}}{2(a+b)}$.

8. Given $\sqrt{9x+4}=3x$ to find x. *Ans.* $x=1\frac{1}{3}$, or $-\frac{1}{3}$.

9. Given $ax^2-bx+c=cx^2+2c$ to find the value of x.

Ans. $x=\dfrac{b\pm\sqrt{b^2+4c(a-c)}}{2(a-c)}$.

10. Given $\dfrac{2c^2}{d^2}+\dfrac{ac}{d}-(a-b)(2c+ad)\dfrac{x}{d}=(a+b)\dfrac{cx}{d}-(a^2-b^2)x^2$ to find the values of x. *Ans.* $x=\dfrac{2c+ad}{d(a+b)}$, or $\dfrac{c}{d(a-b)}$.

11. Given $abx^2+\dfrac{3a^2x}{c}=\dfrac{6a^2+ab-2b^2}{c^2}-\dfrac{b^2x}{c}$ to find the values of x.

Ans. $x=\dfrac{2a-b}{ac}$, or $-\dfrac{3a+2b}{bc}$.

12. Given $80x+\dfrac{3x^2}{4}+\dfrac{21x-27782}{12}=1859\frac{1}{3}-3x^2$ to find the values of x. *Ans.* $x=-46$, or $24\frac{1}{5}$.

(320.) It is frequently advisable to consider several terms as one in the solution of affected quadratics involving radicals.

PROBLEM.

Given $\sqrt{x+12}+\sqrt[4]{x+12}=6$ to find the values of x.

SOLUTION.

If we put $y=\sqrt[4]{x+12}$, y^2 will equal $\sqrt{x+12}$.

$$\therefore \quad y^2+y=6,$$

$$y=\frac{-1\pm5}{2}=2, \text{ or } -3,$$

$$\text{and} \quad y^2=\sqrt{x+12}=4, \text{ or } 9,$$

$$x+12=16, \text{ or } 81,$$

$$x=4, \text{ or } 69.$$

In verifying the value $x=69$, we must take, as the solution indicates, $y=\sqrt[4]{x+12}=-3$.

EXAMPLES.

1. Given $\sqrt{x+10}-\sqrt[4]{x+10}=2$ to find x. *Ans.* $x=6$, or -9.

2. Given $\sqrt{x+21}+\sqrt[4]{x+21}=12$ to find x. *Ans.* $x=60$, or 235.

3. Given $\sqrt{2x+6}+\sqrt[4]{2x+6}=6$ to find x. *Ans.* $x=5$, or $37\frac{1}{2}$.

4. Given $x+\sqrt{x+6}=2+3\sqrt{x+6}$ to find x. *Ans.* $x=10$, or -2.

5. Given $x+5=\sqrt{x+5}+6$ to find x. *Ans.* $x=4$, or -1.

6. Given $x+16-7\sqrt{x+16}=10-4\sqrt{x+16}$ to find x.

Ans. $x=9$, or -12.

7. Given $\sqrt{x+a}+b\sqrt[4]{x+a}=2b^2$ to find of x.

Ans. $x=b^4-a$, or $16b^4-a$.

(321.) It is sometimes advisable to complete the square without reducing the equation to any of the forms given above.

PROBLEM.

Given $\frac{4x^2}{81}+\frac{8x}{18}=8$ to find the values of x.

SOLUTION.

Adding 1 to both members of the equation, and we have

$$\frac{4x^2}{81}+\frac{8x}{18}+1=9.$$

$$\frac{2x}{9}+1=\pm 3.$$

$$\frac{2x}{9}=2, \text{ or } -4.$$

$$2x=18, \text{ or } -36.$$

$$x=9, \text{ or } -18.$$

EXAMPLES.

1. Given $\frac{x^2}{81}-\frac{4x}{18}+5=0$ to find x. *Ans.* $x=9(1\pm 2\sqrt{-1})$.

2. Given $\frac{x^2}{81}-\frac{4x}{81}+\frac{1}{27}=0$ to find x. *Ans.* $x=3$, or 1.

3. Given $\frac{9x^2}{16}+\frac{15x}{4}+4=0$ to find x. *Ans.* $x=-1\frac{1}{3}$, or $-5\frac{1}{3}$.

4. Given $\frac{x^2}{361}-\frac{12x}{19}=-32$ to find x. *Ans.* $x=152$, or 76.

5. Given $\frac{a^2x^2}{b^2}-4ax+3b^2=0$ to find x. *Ans.* $x=\frac{3b^2}{a}$, or $\frac{b^2}{a}$.

6. Given $\frac{4x^2}{9}-4x=7$ to find x. *Ans.* $x=10\frac{1}{2}$, or $-1\frac{1}{2}$.

7. Given $\frac{a^4x^2}{b^3}-\frac{8b^{\frac{1}{2}}x}{a^2}+\frac{12b^4}{a^8}=0$ to find x. *Ans.* $x=\frac{6b^{\frac{7}{2}}}{a^6}$, or $\frac{2b^{\frac{7}{2}}}{a^6}$.

8. Given $\frac{4x^2}{49}+\frac{8x}{21}=6\frac{2}{3}$ to find x. *Ans.* $x=7$, or $-11\frac{2}{3}$.

PROBLEM.

(322.) Given $x^2+x=b$ to find the value of x, b being the product of two consecutive whole numbers.

SOLUTION.

By the conditions, we have by putting a equal to the least of the consecutive numbers $x^2+x=b=a(a+1)$

$$x^2+x=a^2+a.$$

A bare inspection of the last equation shows that one value of x is a, but to get both, we complete the square

$$x^2+x+\tfrac{1}{4}=a^2+a+\tfrac{1}{4}.$$

$$x+\tfrac{1}{2}=\pm(a+\tfrac{1}{2})$$

$$x=a, \text{ or } -(a+1).$$

From which we observe that in an equation of the form

$$x^2+x=b=a(a+1),$$

x has a positive and a negative value, the positive value being equal to the least of the consecutive numbers, and the negative one equal to the other. If the equation were $x^2-x=b=a(a+1)$, the values of x would be found to be the same with opposite signs, namely,

$$x=-a, \text{ or } a+1.$$

SCHOLIUM.—Since, $a^2+a+\frac{1}{4}=a(a+1)+\frac{1}{4}$, we conclude that the product of any two positive consecutive numbers increased by $\frac{1}{4}$ is a perfect square, therefore,

$$2\tfrac{1}{4}=(1\tfrac{1}{2})^2.$$

$$6\tfrac{1}{4}=(2\tfrac{1}{2})^2.$$

$$12\tfrac{1}{4}=(3\tfrac{1}{2})^2.$$

$$20\tfrac{1}{4}=(4\tfrac{1}{2})^2.$$

&c. &c.

PROBLEM.

Given $x^2+x=20$ to find x.

SOLUTION.

$$x^2+x+\tfrac{1}{4}=20\tfrac{1}{4}.$$

$$x+\tfrac{1}{2}=\pm4\tfrac{1}{2}.$$

$$x=4, \text{ or } -5.$$

We might write immediately the value of x thus

$$x=-\tfrac{1}{2}\pm4\tfrac{1}{2}=4, \text{ or } -5,$$

or decide its values by the principle mentioned above.

EXAMPLES.

1. Given $x^2+x=2$ to find the value of x. *Ans.* $x=1$, or -2.

2. Given $x^2-x=6$ to find the value of x. *Ans.* $x=-2$, or 3.

3. Given $x^2+x=12$ to find the value of x. *Ans.* $x=3$, or -4.

4. Given $x^2-x=20$ to find the value of x. *Ans.* $x=-4$, or 5.

5. Given $x^2+x=30$ to find the value of x. *Ans.* $x=5$, or -6.

6. Given $x^2-x=42$ to find the value of x. *Ans.* $x=-6$, or 7.

7. Given $x^2+x=56$ to find the value of x. *Ans.* $x=7$, or -8.

8. Given $x^2-x=72$ to find the value of x. *Ans.* $x=-8$, or 9.

9. Given $x^2+x=90$ to find the value of x. *Ans.* $x=9$, or -10.

10. Given $x^2-x=110$ to find the value of x. *Ans.* $x=-10$, or 11.

11. Given $x^2+x=132$ to find the value of x. *Ans.* $x=11$, or -12.

12. Given $x^2-x=306$ to find the value of x. *Ans.* $x=-17$, or 18.

(323.) The student is not *always* limited to the modes of solution which have already been given.

PROBLEM.

Given $ax^2+bx=c$ to find x.

SOLUTION.

Any two terms can be made a square by adding to them the square of the quotient arising from dividing one of the terms by twice the square root of the other. Hence, ax^2+bx will become a square, if we add to it the square of $\frac{bx}{2x\sqrt{a}}$, or $\frac{b}{2\sqrt{a}}$ which is $\frac{b^2}{4a}$.

Adding $\frac{b^2}{4a}$ to both sides of the given equation, we have

$$ax^2+bx+\frac{b^2}{4a}=c+\frac{b^2}{4a}=\frac{4ac+b^2}{4a}.$$

Extracting square root, we have $x\sqrt{a}+\frac{b}{2\sqrt{a}}=\pm\frac{\sqrt{4ac+b^2}}{2\sqrt{a}}$.

$$2ax+b=\pm\sqrt{4ac+b^2}.$$

$$2ax=-b\pm\sqrt{4ac+b^2}.$$

$$x=\frac{-b\pm\sqrt{4ac+b^2}}{2a}.$$

REMARK.— ax^2+bx will become a square when $\frac{a^2x^2}{4b}$ is added to it. That is, $bx+ax^2+\frac{a^2x^2}{4b}$, or $\frac{a^2x^2}{4b}+ax^2+bx$ is a perfect square. So also, will ax^2+bx become a square when $2x\sqrt{abx}$ is added to it. $2x\sqrt{abx}$ is obtained by considering ax^2, and bx as the first and last term of the square, and finding the middle term which is equal to twice the product of the square roots of the other two. Hence, two terms being given, we can find three terms, any one of which being added to the given terms will produce a square. The first one is the only available one in the solution of quadratics.

EXAMPLES.

1. Given $3x^2+4x=7$ to find x. *Ans.* $x=1$, or $-2\frac{1}{3}$.

2. Given $5x^2+x=22$ to find x. *Ans.* $x=2$, or $-2\frac{1}{5}$.

(324.) Problems of a *special* character may sometimes be solved by modes different from any that have been given. A few are given as a matter of curiosity.

PROBLEM.

Given $3x^2+4x=-1$ (1) to find the value of x.

SOLUTION.

$3x^2+4x+1=0$ (2)=(1) transposed.

If the coefficient of x^2 were 4 instead of 3, the first member of this equation would be a perfect square. In order to render it so, let us add x^2 to both sides, and we have

$$4x^2+4x+1=x^2.$$
$$2x+1=\pm x.$$
$$2x\mp x=-1.$$
$$x \text{ or } 3x=-1.$$
$$x=-1, \text{ or } -\tfrac{1}{3}.$$

EXAMPLES.

1. Given $3x^2+8x=-4$ to find x. *Ans.* $x=-2$, or $-\frac{2}{3}$.

2. Given $15x^2+8x=-1$ to find x. *Ans.* $x=-\frac{1}{3}$, or $-\frac{1}{5}$.

3. Given $15x^2+16x=-4$ to find x. *Ans.* $x=-\frac{2}{3}$, or $-\frac{2}{5}$.

4. Given $12x^2+8x=-1$ to find x, *Ans.* $x=-\frac{1}{2}$, or $-\frac{1}{6}$.

5. Given $12x^2+16x=-4$ to find x. *Ans.* $x=-1$, or $-\frac{1}{3}$.

6. Given $8x^2-12x=-4$ to find x. *Ans.* $x=1$, or $\frac{1}{2}$.

(325.) The student should observe carefully the solution here given of the equation $3x^2+10x=-8$, not because the plan given is the best, but that he may be able to solve in a similar manner the examples which follow, which are intended to give him a foretaste of a principle which is employed in the solution of cubic equations.

PROBLEM.

Given $3x^2+10x=-8$ to find the value of x.

SOLUTION.

$$3x^2+10x+8=0 \qquad (1).$$

Adding 1 to both members of (1), makes the last term of the first member a square.

Thus, $3x^2+10x+9=1 \qquad (2).$

Adding now x^2 to both members of (2), makes the first term of the first member a square.

Thus, $4x^2+10x+9=x^2+1.$

By examining the first member, we see that it would be a perfect square if the middle term were $12x$ instead of $10x$. But we can make it $12x$ by adding $2x$ to both members, which being done, not only renders the first member a perfect square, but also the second one.

$$\text{Thus,}\quad 4x^2+12x+9=x^2+2x+1,$$
$$2x+3=\pm(x+1),$$
$$x=-2, -1\tfrac{1}{3}.$$

EXAMPLES.

1. Given $3x^2+8x=-5$ to find x. *Ans.* $x=-1$, or $-1\frac{2}{3}$.

2. Given $8x^2+10x=-3$ to find x. *Ans.* $x=-\frac{1}{2}$, or $-\frac{3}{4}$.

3. Given $5x^2+8x=-3$ to find x. *Ans.* $x=-\frac{3}{5}$, or -1.

4. Given $3x^2+18x=-15$ to find x. *Ans.* $x=-1$, or -5.

5. Given $12x^2-32x=-5$ to find x. *Ans.* $x=2\frac{1}{2}$, or $\frac{1}{6}$.

6. Given $7x^2-18x=-8$ to find x. *Ans.* $x=2$, or $\frac{4}{7}$.

7. Given $15x^2-16x=7$ to find x. *Ans.* $x=-\frac{1}{3}$, or $1\frac{2}{5}$.

8. Given $15x^2-32x=7$ to find x. *Ans.* $x=2\frac{1}{3}$, or $-\frac{1}{5}$.

9. Given $9x^2+22x=15$ to find x. *Ans.* $x=-3$, or $\frac{5}{9}$.

10. Given $16x^2+8x=8$ to find x. *Ans.* $x=-1$, or $\frac{1}{2}$.

11. Given $24x^2-2x=15$ to find x. *Ans.* $x=-\frac{3}{4}$, or $\frac{5}{6}$.

12. Given $20x^2+10x=24$ to find x. *Ans.* $x=-1\frac{1}{3}$, or $\frac{6}{7}$.

MISCELLANEOUS EXAMPLES.

1. Given $\frac{x^2}{6}-1=x+11$ to find x. *Ans.* $x=12$, or -6.

2. Given $\frac{6}{x+1}+\frac{2}{x}=3$ to find x. *Ans.* $x=2$, or $-\frac{1}{3}$.

3. Given $6x+\frac{35-3x}{x}=44$ to find x. *Ans.* $x=7$, or $\frac{5}{6}$.

4. Given $\frac{16}{x}-\frac{100-9x}{4x^2}=3$ to find x. *Ans.* $x=4$, or $2\frac{1}{12}$.

5. Given $\frac{a-x}{x}+\frac{x}{a-x}=\frac{b}{c}$ to find x.

Ans. $x=a\left(\frac{1}{2}\mp\frac{1}{2(2c+b)}\sqrt{b^2-4c^2}\right)$.

6. Given $\frac{\sqrt{4x}+2}{4+\sqrt{x}}=\frac{4-\sqrt{x}}{\sqrt{x}}$ to find x. *Ans.* $x=4$, or $\frac{64}{9}$.

7. Given $\frac{\sqrt{a^2x}+b}{a+\sqrt{x}}=\frac{a-\sqrt{x}}{\sqrt{x}}$ to find x.

Ans. $x=\left(\frac{-b\pm\sqrt{4a^3+4a^2+b^2}}{2(a+1)}\right)^2$.

8. Given $(\sqrt{x+5})(\sqrt{x+12})=12$ to find x.

Ans. $x=4$, or -21.

9. Given $\frac{8-x}{2}-\frac{2x-11}{x-3}=\frac{x-2}{6}$ to find x. *Ans.* $x=6$, or $\frac{1}{2}$.

10. Given $\frac{x}{2}-\frac{x^{\frac{1}{2}}}{3}=22\frac{1}{6}$ to find x. *Ans.* $x=49$, or $\frac{361}{9}$.

11. Given $\sqrt{2x+1}+2\sqrt{x}=\frac{21}{\sqrt{2x+1}}$ to find x.

Ans. $x=4$, or -25.

12. Given $\sqrt{x^5}+\sqrt{x^3}=6\sqrt{x}$ to find x. *Ans.* $x=2$, or -3.

13. Given $\sqrt{x^3}-2\sqrt{x}-x=0$ to find x. *Ans.* $x=4$, or 1.

14. Given $\sqrt[3]{x^3-a^3}=x-b$ to find x.

Ans. $x=\frac{b}{2}\pm\frac{1}{12b}\sqrt{48a^3b-12b^4}$.

15. Given $\frac{\frac{3}{5}\sqrt{x}-2}{x-5}=\frac{1}{20}$ to find x. *Ans.* $x=49$, or 25.

16. Given $x+\sqrt{5x+10}=8$ to find x. *Ans.* $x=18$, or 3.

17. Given $x+\sqrt{10x+6}=9$ to find x. *Ans.* $x=25$, or 3.

18. Given $2\sqrt{x-a}+3\sqrt{2x}=\dfrac{7a+5x}{\sqrt{x-a}}$ to find x.

Ans. $x=9a$, or $-a$.

19. Given $3x^2-x=140$ to find x. *Ans.* $x=7$, or $-6\frac{2}{3}$.

20. Given $5x^2+\dfrac{7x}{2}=7x^2-51$ to find x. *Ans.* $x=6$, or $-4\frac{1}{4}$.

21. Given $2x^2-\dfrac{4x-4}{3}=7x$ to find x. *Ans.* $x=4$, or $\frac{1}{6}$.

22. Given $3x^2-17x=2x^2+84$ to find x. *Ans.* $x=21$, or -4.

23. Given $x^2-x+3=45$ to find x. *Ans.* $x=7$, or -6.

24. Given $4x-\dfrac{14-x}{x+1}=14$ to find x. *Ans.* $x=4$, or $-1\frac{3}{4}$.

25. Given $\dfrac{10}{x}-\dfrac{14-2x}{x^2}=\dfrac{22}{9}$ to find x. *Ans.* $x=3$, or $1\frac{10}{11}$.

26. Given $5x^2-4x+3=159$ to find x. *Ans.* $x=6$, or $-5\frac{1}{5}$.

27. Given $3x-\dfrac{1121-4x}{x}=2$ to find x. *Ans.* $x=19$, or $-19\frac{2}{3}$.

28. Given $\dfrac{8-x}{2}-\dfrac{2x-11}{x-3}=\dfrac{x-2}{6}$ to find x. *Ans.* $x=6$, or $\frac{1}{2}$.

29. Given $5x-\dfrac{3x-3}{x-3}=2x+\dfrac{3x-6}{2}$ to find x. *Ans.* $x=4$, or -1.

30. Given $3x-\dfrac{169-3x}{x}=29$ to find x. *Ans.* $x=13$, or $-4\frac{1}{3}$.

31. Given $16-\dfrac{2x^2}{3}=\dfrac{4x}{5}+7\frac{3}{5}$ to find x. *Ans.* $x=3$, or $-4\frac{1}{5}$.

32. Given $\dfrac{3x-4}{x-4}+1=10-\dfrac{x-2}{2}$ to find x. *Ans.* $x=12$, or 6.

33. Given $\dfrac{3x+4}{5}-\dfrac{30-2x}{x-6}=\dfrac{7x-14}{10}$ to find x.

Ans. $x=36$, or 12.

34. Given $3x-\dfrac{3x-10}{9-2x}=2+\dfrac{6x^2-40}{2x-1}$ to find x.

Ans. $x=11\frac{1}{2}$, or 4.

35. Given $\frac{x}{5+x}+\frac{7}{6-4x}=\frac{11x}{11x-8}$ to find x.

Ans. $x=1$, or $-\frac{40}{47}$.

36. Given $\frac{90}{x}-\frac{27}{x+2}=\frac{90}{x+1}$ to find x. *Ans.* $x=4$, or $-1\frac{2}{3}$.

37. Given $\frac{1}{x^2-3x}+\frac{1}{x^2+4x}=\frac{9}{8x}$ to find x. *Ans.* $x=4$, or $-3\frac{2}{9}$.

38. Given $\frac{x^3-10x^2+1}{x^2-6x+9}=x-3$ to find x. *Ans.* $x=1$, or -28.

39. Given $\frac{x}{7-x}+\frac{7-x}{x}=2\frac{9}{10}$ to find x. *Ans.* $x=5$, or 2.

40. Given $\frac{7-12x}{x^{\frac{3}{2}}}=\frac{x}{\sqrt{x}}-\frac{8x+110}{\sqrt{x^3}}$ to find x.

Ans. $x=9$, or -13.

41. Given $\frac{a-x}{x}+\frac{x}{a-x}=\frac{b}{c}$ to find x. *Ans.* $x=\frac{2ac}{2c+b\pm\sqrt{b^2-4c^2}}$.

42. Given $\sqrt{x+60}+\sqrt{x^2+9}=\frac{2\sqrt{x^3+60x^2+9x+540}+89}{\sqrt{x+60}+\sqrt{x^2+9}}$ to find x.

Ans. $x=4$, or -5.

43. Given $\frac{123+41\sqrt{x}}{5\sqrt{x}-x}=\frac{20\sqrt{x}+4x}{3-\sqrt{x}}-\frac{2x^2}{(5\sqrt{x}-x)(3-\sqrt{x})}$ to find x.

Ans. $x=20\frac{1}{2}$, or 3.

44. Given $\frac{x}{\sqrt{x}+\sqrt{a-x}}+\frac{x}{\sqrt{x}-\sqrt{a-x}}=\frac{b}{\sqrt{x}}$ to find x.

Ans. $x=\frac{b\pm\sqrt{b^2-2ab}}{2}$.

45. Given $\frac{175x-350}{x}+10x=195$ to find x. *Ans.* $x=7$, or -5.

46. Given $\frac{x^2}{2}-\frac{x}{3}+20\frac{1}{2}=42\frac{2}{3}$ to find x. *Ans.* $x=7$, or $-6\frac{1}{3}$.

47. Given $\frac{2x+\sqrt{x}}{2x-\sqrt{x}}=3\frac{7}{15}-3.\frac{2x-\sqrt{x}}{2x+\sqrt{x}}$ to find x. *Ans.* $x=4$, or $3\frac{1}{16}$,

48. Given $\frac{54-9\sqrt{x}}{x+2\sqrt{x}}=\frac{23x-46\sqrt{x}}{6+\sqrt{x}}+\frac{7x^2-3x+4}{(x+2\sqrt{x})(6+\sqrt{x})}$ to find x.

Ans. $x=5$, or $-2\frac{2}{15}$.

49. Given $\frac{16-4\sqrt{x}}{8-3\sqrt{x}}=\frac{88+33\sqrt{x}}{4+\sqrt{x}}+\frac{x^2-5x+11}{(8-3\sqrt{x})(4+\sqrt{x})}$ to find x.

Ans. $x=93$, or 7.

50. Given $\frac{x-\sqrt{x+1}}{x+\sqrt{x+1}}=\frac{5}{11}$ to find x. *Ans.* $x=8$, or $-\frac{8}{9}$.

51. Given $(\sqrt{4x+5})(\sqrt{7x+1})=30$ to find x.

Ans. $x=5$, or $-\frac{179}{28}$.

52. Given $\frac{15x-5}{1+5\sqrt{x}}+\frac{2}{\sqrt{x}}=3\sqrt{x}$ to find x. *Ans.* $x=4$, or $\frac{1}{9}$.

53. Given $9a^4b^4x^2-6a^3b^2x=b^2$ to find x. *Ans.* $x=\frac{a^2\pm\sqrt{a^2+b^2}}{3a^2b^2}$.

54. Given $3\sqrt{112-8x}=19+\sqrt{3x+7}$ to find x.

Ans. $x=6$, or $\frac{7398}{625}$.

55. Given $\sqrt{2x+7}+\sqrt{3x-18}=\sqrt{7x+1}$ to find x.

Ans. $x=9$, or $-3\frac{3}{5}$.

56. Given $\frac{a^2x^2}{b^2}-\frac{2ax}{c}+\frac{d^2}{c^2}=0$ to find x. *Ans.* $x=\frac{b(b\pm\sqrt{b^2-d^2})}{a\quad c}$.

57. Given $x+\sqrt{x}:x-\sqrt{x}::3\sqrt{x}+6:2\sqrt{x}$ to find x.

Ans. $x=9$, or 4.

58. Given $\sqrt{(4+x)}(5-x)=2x-10$ to find x. *Ans.* $x=5$, or $3\frac{1}{5}$.

59. Given $x^2-x-40=170$ to find x. *Ans.* $x=15$, or -14.

60. Given $x+\frac{\sqrt{x-3}}{2}=8$ to find x. *Ans.* $x=7$, or $9\frac{1}{4}$.

61. Given $2\sqrt[3]{x^2}+3\sqrt[3]{x}=2$ to find x. *Ans.* $x=\frac{1}{8}$, or -8.

62. Given $4x=\frac{36-x}{x}+46$ to find x. *Ans.* $x=12$, or $-\frac{3}{4}$,

63. Given $\frac{7+x}{7-x}+\frac{7-x}{7+x}=\frac{29}{10}$ to find x. *Ans.* $x=7$, or -3.

64. Given $\frac{3x+5}{3x-5}-\frac{3x-5}{3x+5}=\frac{135}{176}$ to find x. *Ans.* $x=9$, or $-\frac{25}{81}$.

65. Given $\sqrt{(x-1)(x-2)}+\sqrt{(x-3)(x-4)}=\sqrt{2}$ to find x.

Ans. $x=3$, or 2.

66. Given $\frac{2}{(x+2)^{\frac{3}{2}}}+\frac{\sqrt{x+2}}{2}=\frac{17}{4\sqrt{x+2}}$ to find x.

Ans. $x=6$, or $-\frac{3}{2}$.

67. Given $7\sqrt{\frac{3x}{2}-5}-\sqrt{\frac{x}{5}+45}=\frac{7}{4}\sqrt{10x+56}$ to find x.

Ans. $x=20$, or $\frac{1456898 0}{2874649}$.

68. Given $ab^3x^2+(1+c)bd\sqrt{c}+cb^2x^2=[b^3d\sqrt{c}+(ab+c)(1+c)]x$ to find x.

Ans. $x=\frac{bd\sqrt{c}}{ab+c}$, or $\frac{1+c}{b^2}$.

69. Given $\frac{x}{x+60}=\frac{7}{3x-5}$ to find x. *Ans.* $x=14$, or -10.

70. Given $\frac{40}{x-5}+\frac{27}{x}=13$ to find x. *Ans.* $x=9$, or $1\frac{2}{13}$.

71. Given $\frac{8x}{x+2}-6=\frac{20}{3x}$ to find x. *Ans.* $x=10$, or $-\frac{2}{3}$.

72. Given $\frac{48}{x+3}=\frac{165}{x+10}-5$ to find x. *Ans.* $x=5\frac{2}{5}$, or 5.

73. Given $\frac{31}{6x}=\frac{16}{117-2x}+1$ to find x. *Ans.* $x=67\frac{1}{6}$, or $4\frac{1}{2}$.

74. Given $\frac{2x+3}{10-x}=\frac{2x}{25-3x}-6\frac{1}{2}$ to find x. *Ans.* $x=13\frac{22}{31}$, or 8.

75. Given $\frac{25x+180}{10x-81}=\frac{40x}{5x-8}-\frac{3}{5}$ to find x.

Ans. $x=14\frac{3}{5}$, or $\frac{72}{245}$.

76. Given $\frac{18+x}{6(3-x)}=\frac{20x+9}{19-7x}-\frac{65}{4(3-x)}$ to find x.

Ans. $x=7\frac{22}{113}$, or $2\frac{1}{2}$.

77. Given $\frac{5a+10ab^2}{9b^2-3a^2b^2}x^2-\left(\frac{5\sqrt{a+b}}{3b^3}+\frac{(1+2b^2)cd\sqrt{c}}{3-a^2}\right)x+\frac{cd}{ab}\sqrt{(a+b)c}=0$ to find x.

Ans. $x=\frac{(3-a^2)\sqrt{a+b}}{ab(1+2b^2)}$, or $\frac{3b^2cd\sqrt{c}}{5a}$.

AFFECTED QUADRATICS INVOLVING TWO UNKNOWN QUANTITIES.

(326.) In solving equations of this character the usual plans of elimination may be employed. The student should adopt that plan which seems to be best for the example under consideration.

EXAMPLES.

1. Given $\left\{\begin{array}{c} x-y=15, \\ \dfrac{xy}{2}=y^3, \end{array}\right\}$ to find the values of x and y.

Ans. $\left\{\begin{array}{l} x=18, \text{ or } \frac{25}{2}, \\ y=3, \text{ or } -\frac{5}{2}. \end{array}\right.$

2. Given $\left\{\begin{array}{l} \dfrac{10x+y}{xy}=3, \\ 9y-9x=18, \end{array}\right\}$ to find the values of x and y.

Ans. $\left\{\begin{array}{l} x=2, \text{ or } -\frac{1}{3}, \\ y=4, \text{ or } \frac{5}{3}. \end{array}\right.$

3. Given $\left\{\begin{array}{l} x+y : x-y :: 13 : 5, \\ y^2+x=25, \end{array}\right\}$ to find the values of x and y.

Ans. $\left\{\begin{array}{l} x=9, \text{ or } -\frac{225}{16}, \\ y=4, \text{ or } -\frac{25}{4}. \end{array}\right.$

4. Given $\left\{\begin{array}{l} x-x^{\frac{1}{2}}=3-y, \\ 4-x\ =y-y^{\frac{1}{2}}, \end{array}\right\}$ to find the values of x and y.

Ans. $\left\{\begin{array}{l} x=4, \text{ or } \frac{1}{4}, \\ y=1, \text{ or } \frac{9}{4}. \end{array}\right.$

5. Given $\left\{\begin{array}{l} x^{\frac{2}{3}}y^{\frac{3}{2}}=2y^2, \\ 8x^{\frac{1}{3}}-y^{\frac{1}{2}}=14, \end{array}\right\}$ to find the values of x and y.

Ans. $\left\{\begin{array}{l} x=(14)^3, \text{ or } 8, \\ y=(98)^2, \text{ or } 4. \end{array}\right.$

6. Given $\left\{\begin{array}{l} x^{\frac{3}{2}}+y^{\frac{2}{3}}=3x, \\ x^{\frac{1}{2}}+y^{\frac{1}{3}}=x, \end{array}\right\}$ to find the values of x and y.

Ans. $\left\{\begin{array}{l} x=4, \text{ or } 1, \\ y=8, \text{ or } 0. \end{array}\right.$

7. Given $\left\{\begin{array}{c} x+y+\sqrt{y}=2\sqrt{xy}+\sqrt{x}, \\ \sqrt{x}+\sqrt{y}=5, \end{array}\right\}$ to find the values of x and y.

Ans. $\left\{\begin{array}{l} x=9, \text{ or } \frac{25}{4}, \\ y=4, \text{ or } \frac{25}{4}. \end{array}\right.$

8. Given $\left\{\begin{array}{l} \dfrac{x^2}{y^2}+\dfrac{4x}{y}=\dfrac{85}{9}, \\ x-\ y=2, \end{array}\right\}$ to find the values of x and y.

Ans. $\left\{\begin{array}{l} x=5, \text{ or } \frac{17}{10}, \\ y=3, \text{ or } -\frac{3}{10}. \end{array}\right.$

9. Given $\left\{\begin{array}{l} \sqrt{y}+\sqrt{x}:\sqrt{y}-\sqrt{x}::\sqrt{x}+2:1, \\ \dfrac{\sqrt{y}+2}{\sqrt{x}}-1=\dfrac{3\sqrt{x}+1+\dfrac{\sqrt{y}}{\sqrt{x}}}{\sqrt{y}} \end{array}\right\}$ to find x and y.

Ans. $\left\{\begin{array}{l} x=1, \text{ or } \frac{1}{9}, \\ y=4, \text{ or } \frac{16}{9}. \end{array}\right.$

10. Given $\left\{\begin{array}{l} x+4y=14, \\ y^2+4x=2y+11 \end{array}\right\}$ to find the values of x and y.

Ans. $\left\{\begin{array}{l} x=-46, \text{ or } 2, \\ y=15, \text{ or } 3. \end{array}\right.$

11. Given $\left\{\begin{array}{l} 2x+3y=118, \\ 5x^2-7y^2=4333, \end{array}\right\}$ to find the values of x and y.

Ans. $\left\{\begin{array}{l} x=35, \text{ or } -\frac{3899}{17}, \\ y=16, \text{ or } \frac{3268}{17}. \end{array}\right.$

12. Given $\left\{\begin{array}{l} \dfrac{2x+7y}{4x}=2y-\dfrac{51+2x}{10}, \\ \dfrac{4x+3y}{16}=y-2, \end{array}\right\}$ to find the values of x and y.

Ans. $\left\{\begin{array}{l} x=5, \text{ or } -\frac{56}{27}, \\ y=4, \text{ or } \frac{640}{351}, \end{array}\right.$

13. Given $\left\{\begin{array}{l} \dfrac{4xy+3y-3}{5x}-1=\dfrac{4y+3x-2}{5}-\dfrac{18-x}{3}, \\ \dfrac{3x+y}{7}=\dfrac{3x-5y}{3}+2, \end{array}\right\}$ to find x and y.

Ans. $\left\{\begin{array}{l} x=6, \text{ or } -\frac{3}{266}, \\ y=3, \text{ or } \frac{2784}{2527}. \end{array}\right.$

14. Given $\left\{\begin{array}{l} x+4\sqrt{x}+4y=21+8\sqrt{y}+4\sqrt{xy}, \\ \sqrt{x}+\sqrt{y}=6, \end{array}\right\}$ to find x and y.

Ans. $\left\{\begin{array}{l} x=25, \text{ or } \frac{25}{9}, \\ y=1, \text{ or } \frac{169}{9}. \end{array}\right.$

327. QUESTIONS PRODUCING AFFECTED QUADRATICS.

QUESTION.

A and *B* sold 130 yards of calico (of which 40 yards were *A*'s, and 90 *B*'s), for \$42. Now, *A* sold for \$1, $\frac{1}{3}$ of a yard more than *B* did. How many yards did each sell for \$1 ?

Let $x =$ the No. of yards B sold for \$1.

$\therefore x+\frac{1}{3} =$ " " A " "

$\frac{1}{x} =$ what B got for 1 yard.

$\frac{1}{x+\frac{1}{3}} =$ " A " "

$\frac{90}{x} =$ " B " 90 yards.

$\frac{40}{x+\frac{1}{3}} =$ " A " 40 "

$$\therefore\ 42=\frac{90}{x}+\frac{40}{x+\frac{1}{3}}$$

$$21=\frac{45}{x}+\frac{20}{x+\frac{1}{3}}$$

$$21x^2+7x=45x+15+20x$$

$$21x^2-58x=15$$

$x=\frac{29\pm34}{21}=3$, or $-\frac{5}{21}$ No. of yards B sold for \$1.

$x+\frac{1}{3}=3\frac{1}{3}$ " " A " "

QUESTIONS.

1. A merchant sold a quantity of cloth for \$39, and gained as much *per* cent. as the cloth cost him. What was the price of the cloth? *Ans.* \$30.

2. There are two numbers whose difference is 9, and their sum multiplied by the greater produces 266. What are the numbers? *Ans.* 14 and 5, or $-9\frac{1}{2}$ and $-18\frac{1}{2}$.

3. It is required to find two numbers, the first of which may be to the second as the second is to 16; and the sum of the squares of the numbers may be equal to 225. *Ans.* 9 and 12.

4. Bought two sorts of linen for 6 crowns. An ell of the finer cost as many shillings as there were ells of the finer. Also, 28 ells of the coarser (which was the whole quantity) were at such a price that 8 ells cost as many shillings as 1 ell of the finer. How many ells were there of the finer, and what was the value of each piece?

Ans. 4 ells of the finer; the value of the finer 16 shillings, and of the coarser 14 shillings.

5. Two partners, A and B, gained \$18 by trade. A's money was in trade 12 months, and he received for his principal and gain \$26. B's money, which was \$30, was in trade 16 months. How much did A put in trade? *Ans.* \$20.

6. A person bought some sheep for \$72, and found that if he had bought 6 more for the same money, he would have paid \$1 less for each. How many did he buy, and what was the price of each?

Ans. 18 sheep, at \$4 a piece.

7. The plate of a looking-glass is 12 inches by 18, and is to be framed with a frame of equal width, whose area is to be equal to that of the glass. What is the width of the frame? *Ans.* 3 inches.

8. There are two square buildings, that are paved with stones, a foot square each. The side of one building exceeds that of the other by 12 feet, and both their pavements taken together contain 2120 stones. What are the lengths of them separately?

Ans. 26, and 38 feet, respectively.

9. A laborer dug two trenches, one of which was 6 yards longer than the other, for £17 16s., and the digging of each of them cost as many shillings a yard as there were yards in length. What was the length of each? *Ans.* 10, and 16 yards.

10. A company at a tavern had £8 15s. to pay; but before the bill was paid, 2 of them sneaked off, in consequence of which those that remained had each 10 shillings more to pay. How many were in the company at first? *Ans.* 7.

11. A grazier bought as many sheep as cost him £60; out of which he reserved 15, and sold the remainder for £54, gaining 2 shillings a head on those he sold. How many sheep did he buy, and what was the price of each? *Ans.* 75 sheep, at 16 shillings each.

12. What two numbers are those whose sum is 19, and whose difference multiplied by the greater is 60? *Ans.* 12 and 7.

13. If the square of a certain number be taken from 40, and the square root of this difference be increased by 10, and the sum multiplied by 2, and the product divided by the number itself, the quotient will be 4. What is the number? *Ans.* 6.

14. A person being asked his age, answered, if you add the square root of it to $\frac{1}{2}$ of it, and subtract 12, there will remain nothing. How old was he? *Ans.* 16, or 36. (Prove the last answer.)

15. A and B set out from two towns which were at the distance of

247 miles, and traveled the direct road till they met. *A* went 9 miles a day; and the number of days, at the end of which they met, was greater by 3 than the number of miles which *B* went in a day. How many miles did each go? *Ans.* *A* 117, and *B* 130.

16. *A* set out from *C* toward *D*, and traveled 7 miles a day. After he had gone 32 miles, *B* set out from *D* toward *C*, and went every day $\frac{1}{19}$ of the whole journey; and after he had traveled as many days as he went miles in one day, he met *A*. What is the distance from *C* to *D*? *Ans.* 152, or 76.

17. Three merchants, *A*, *B*, and *C*, made a joint stock, by which they gained a sum less than that by \$80. *A*'s share of the gain was \$60; and his contribution to the stock was \$17 more than *B*'s. Also, *B* and *C* together contributed \$325. How much did each contribute? *Ans.* *A* \$75, *B* \$58, *C* \$267.

18. The joint stock of two partners, *A* and *B*, was \$416. *A*'s money was in trade 9 months, and *B*'s 6 months; when they shared stock and gain, *A* received \$228, and *B* \$252. How much was each man's stock? *Ans.* *A*'s \$192, and *B*'s \$224.

19. A body of men were formed into a hollow square 3 deep, when it was observed, that with an addition of 25 to their number, a solid square might be formed, of which the number of men in each side would be greater by 22 than the square root of the number of men in each side of the hollow square. What was the number of men in the hollow square? *Ans.* 936.

20. A merchant bought a number of pieces of two different kinds of silk for £92 3s. There were as many pieces bought of each kind, and as many shillings paid per yard for them, as a piece of that kind contained yards. Now 2 pieces, one of each kind, together measured 19 yards. How many yards were there in each? *Ans.* 11 and 8.

21. A vintner sold 7 dozen sherry and 12 claret, for £50. He sold 3 dozen more of sherry for £10 than he did of claret for £6. What was the price of each per dozen?

Ans. Sherry £2, and claret £3.

22. *A* and *B* hired a pasture, into which *A* put 4 horses, and *B* as many as cost him 18 shillings a week. Afterward *B* put in 2 additional horses, and found that he must pay 20 shillings a week. At what rate was the pasture hired? *Ans.* 30 shillings per week.

23. A merchant bought 54 gallons Cognac brandy, and a certain

quantity of British. For the former he gave $\frac{1}{2}$ as many shillings per gallon as there were gallons of British, and for the latter 4 shillings per gallon less. He sold the mixture at 10 shillings per gallon, and lost £28 16s. by his bargain. What was the price of the Cognac, and the number of gallons of British?

Ans. Cognac 18s. per gallon, and 36 gallons of British.

24. What number is that, which being divided by the product of its two digits, the quotient is 2, and if 27 be added to it, the digits will be inverted? *Ans.* 36.

25. I have a certain number in my mind; this I multiply by $2\frac{1}{3}$, add 7 to the product, and multiply this sum by 8 times the number; I now divide by 14, and subtract from the quotient 4 times the number, and obtain 2352. What number is it? *Ans.* 42.

26. What two numbers are those whose sum is 100, and the sum of whose square roots is 14? *Ans.* 64 and 36.

27. What number is that which if you subtract from 10 and multiply the remainder by the number itself, the product shall be 21?

Ans. 7, or 3.

28. What are the two parts of 24 whose product is equal to 35 times their difference? *Ans.* 10 and 14.

29. What two numbers are those whose sum is 8, and the sum of whose cubes is 152? *Ans.* 3 and 5.

30. Two partners, *A* and *B*, gained \$140 by trade; *A*'s money was 3 months in trade, and his gain was \$60 less than his stock, and *B*'s money, which was \$50 more than *A*'s, was in trade 5 months. What was *A*'s stock? *Ans.* \$100.

31. What two numbers are those, the difference of whose squares is q^2, and which being multiplied, respectively, by a and b, the difference of the products is s^2?

$$\textit{Ans.}\ \frac{as^2 \pm b\sqrt{s^4-(a^2-b^2)q^2}}{a^2-b^2}.$$

$$\text{and}\ \frac{bs^2 \pm a\sqrt{s^4-(a^2-b^2)q^2}}{a^2-b^2}.$$

32. Into what two parts can a and b each be divided, such that the

product of one part of a by one part of b shall be p, and the product of the remaining parts q?

$$\text{Ans.}\ \begin{cases} a = \dfrac{ab-(q-p)\pm\sqrt{\{ab-(q-p)\}^2-4abp}}{2b}. \\[2ex] \quad + \dfrac{ab+(q-p)\mp\sqrt{\{ab-(q-p)\}^2-4abp}}{2b}. \\[2ex] b = \dfrac{ab-(q-p)\pm\sqrt{\{ab-(q-p)\}^2-4abp}}{2a}. \\[2ex] \quad + \dfrac{ab+(q-p)\pm\sqrt{\{ab-(q-p)\}^2-4abp}}{2a}. \end{cases}$$

33. During the time that the shadow of a sundial, which shows true time, moves from one o'clock to five, a clock which is too fast a certain number of hours and minutes, strikes a number of strokes = that number of hours and minutes, and it is observed that the number of minutes is less by 41 than the square of the number which the clock strikes at the last time of striking. The clock does not strike twelve during the time. How much is it too fast?

Ans. 3 hours and 23 minutes.

34. *A* and *B* engage to reap a field for £4 10*s.*; and as *A*, alone, could reap it in 9 days, they promise to complete it in 5 days. They found, however, that they were obliged to call in *C*, an inferior workman, to assist them for the 2 last days, in consequence of which *B* received 3*s.* 9*d.* less than he otherwise would have done. In what time could *B*, or *C*, alone, reap the field?

Ans. *B* in 15, and *C* in 18 days.

35. There are three numbers, the difference of whose differences is 8; their sum is 41; and the sum of their squares 699. What are the numbers? *Ans.* 7, 11, and 23.

36. There are three numbers, the difference of whose differences is 5; their sum is 44; and their continued product is 1950. What are the numbers? *Ans.* 6, 13, and 25.

37. A grocer sold 80 pounds of mace, and 100 pounds of cloves for \$65; but he sold 60 pounds more of cloves for \$20 than he did of mace for \$10. What was the price of a pound of each?

Ans. Mace 50, and cloves 25 cents a pound.

38. The fore-wheel of a carriage makes 6 revolution more than the hind-wheel in going 120 yards; but if the periphery of each wheel

be increased one yard, it will make only 4 revolutions more than the hind-wheel in the same space. What is the circumference of each?

Ans. 4 and 5 yards.

39. *A* and *B* were going to market, the first with cucumbers, and the second with 3 times as many eggs; and they find that if *B* gives all his eggs for the cucumbers, *A* would lose 10 pence, according to the rate at which they were selling. *A*, therefore, reserves $\frac{2}{5}$ of his cucumbers, by which *B* would lose sixpence, according to the same rate. But *B*, selling the cucumbers at sixpence apiece, gains upon the whole the price of 6 eggs. What was the number of eggs, and cucumbers, and the price of one of each?

Ans. 30 eggs at 1 penny a piece, and 10 cucumbers at 4 pence apiece.

40. A person bought a certain number of larks and sparrows for 6 shillings. He gave as many pence per dozen for larks as there were sparrows, and as many pence per score for sparrows as there were larks. If he had bought 10 more of each (the price of the larks remaining the same), and had given as much per dozen for sparrows as he gave per score for larks, they would have cost £1 5*s.* 5*d.* What was the number of each? *Ans.* 15 larks, and 36 sparrows.

41. A poulterer bought a certain number of ducks and 18 turkeys for $110: each turkey costing within 1 dollar as much as three ducks. He afterwards bought as many ducks and 5 more, and 20 turkeys, giving one dollar a piece more for each duck and turkey than before; and found that the value of his former purchase was to the value of the latter one : : 2 : 3. What was the number of ducks, and the prices of the ducks and turkeys at the first purchase?

Ans. 10 ducks, and the price of a duck $2, and of a turkey $5.

42. *A* and *B* put out different sums at interest, amounting together to $200. *B*'s rate of interest was 1 per cent. more than *A*'s. At the end of 5 years, *B*'s accumulated simple interest wanted but $4 to be double of *A*'s. At the end of 10 years, *A*'s principal and interest was to *B*'s as 5 : 8. What was the sums put out by each, and the rate per cent.?

Ans. *A* put out $80 at 5 per cent., and *B* $120 at 6 per cent.

43. When the price of brandy was 3 times the price of British spirit, a merchant made two mixtures of brandy and British spirit, and the prices per gallon were in the ratio of 9 to 10. He afterwards mixed twice as much brandy with the same quantity of British spirit

in each case, and the relative prices were the same as before. What was the ratio of the quantities mixed?

Ans. The first mixtures were as 3 to 1, and 2 to 1, and the second as 3 to 2, and 1 to 1.

44. *A* and *B* traveled on the same road and at the same rate from Huntingdon to London. At the 50th milestone from London, *A* overtook a drove of geese which were proceeding at the rate of 3 miles in 2 hours, and 2 hours afterwards met a stage-wagon, which was moving at the rate of 9 miles in 4 hours. *B* overtook the same drove of geese at the 45th milestone, and met the same stage-wagon exactly 40 minutes before he came to the 31st milestone. Where was *B* when *A* reached London? *Ans.* 25 miles from London.

45. The difference between the hypothenuse and base of a right-angled triangle is 6, and the difference between the hypothenuse and the perpendicular is 3. What are the sides? *Ans.* 15, 12, and 9.

46. In digging among some ruins, the workmen found 9 urns, together containing 60 gold coins; the 2d and 8th containing 8 and 4 respectively. They secreted a certain number of these, greater than the number they left; which being afterwards recovered, it was found that the number of urns secreted was to the number left as the number of coins secreted was to the number remaining. Now if instead of taking the 2d urn, they had carried off the 8th, then the number of coins taken away would have been to the number remaining as the square of the number of urns secreted to the difference between that square and 20 times the number of urns remaining. What number of each was secreted?

Ans. 6 urns and 40 coins.

47. Bacchus caught Silenus asleep by the side of a full cask, and seized the opportunity of drinking, which he continued for $\frac{2}{3}$ of the time that Silenus would have taken to empty the whole cask. After that Silenus awoke, and drank what Bacchus had left. Had they drank both together, it would have been emptied 2 hours sooner, and Bacchus would have drank only $\frac{1}{2}$ what he left Silenus. How long would it have taken each to have emptied the cask separately?

Ans. Silenus in 3 hours, and Bacchus in 6.

CHAPTER XII.

CUBIC EQUATIONS.

(328.) Cubic Equations may be divided into

Pure Cubics,
Incomplete Cubics, and
Affected Cubics.

(239.) *A pure cubic equation* is one in which the unknown quantity is contained in but one term, and whose exponent is 3, or a fraction which when reduced to its lowest terms the numerator is 3; as,

$$ax^3=\pm b;\ ax^{\frac{3}{2}}=\pm b;\ ax^{\frac{6}{8}}=\pm b, \text{ \&c.}$$

(330.) The solution of a pure cubic presents no difficulty.

PROBLEM.

Given $3x^3=24$ to find x.

SOLUTION.

$$3x^3=24,$$
$$x^3=8,$$
$$x^3-8=0,$$
$$(x-2)(x^2+2x+4)=0.$$

Whence we get $x-2=0$,
or $x^2+2x+4=0$,
whose solutions give $x=2$, or $-1\pm\sqrt{-3}$.

If but one value of the unknown quantity were required, we might in the equation $x^3=8$ merely take the cube root of both sides, and obtain $x=2$.

EXAMPLES.

1. Given $x^3=27$ to find x. *Ans.* $x=3$, or $\dfrac{-3\pm3\sqrt{-3}}{2}$.

2. Given $\frac{1}{2}x^3=32$ to find x. *Ans.* $x=4$, or $-2\pm2\sqrt{-3}$.

3. Given $x^3=4$ to find x.

Ans. $x=\sqrt[3]{4}$, or $-\frac{1}{2}\sqrt[3]{4}\pm\sqrt{-\frac{3}{2}\sqrt[3]{2}}$.

4. Given $\left\{\begin{array}{l} x+y:x-y::3:1 \\ x^3-y^3=56 \end{array}\right\}$ to find the values of x and y.

Ans. $\left\{\begin{array}{l} x=4, \text{ or } -2\pm2\sqrt{-3} \\ y=2, \text{ or } -1\pm\sqrt{-3}. \end{array}\right.$

5. Given $\left\{\begin{array}{l} x-y:x::5:6 \\ xy^2=384 \end{array}\right\}$ to find the values of x and y.

Ans. $\left\{\begin{array}{l} x=24, \text{ or } -12\pm12\sqrt{-3} \\ y=4, \text{ or } -2\pm2\sqrt{-3} \end{array}\right.$.

(331.) An *incomplete cubic equation* is one in which the unknown quantity is contained in but two terms: the exponent of one being 3, and the other 1 or 2; or, the exponent of one being 3 times a proper fraction whose numerator is 1, and which is either the exponent or half the exponent of the other term; as,

$ax^3\pm bx=\pm c$; $ax^3\pm bx^2=\pm c$; $ax\pm bx^{\frac{1}{3}}=\pm c$; $ax\pm bx^{\frac{2}{3}}=\pm c$; &c.

REMARK.—No general practicable method of obtaining the values of the unknown quantity in incomplete or affected cubics has as yet been discovered, and we are, therefore, compelled to resort to what are called *special, tentative,* or *approximative methods.*

Some of these methods are here set forth.

PROBLEM.

Given $x^3+3x=14$ to find the values of x.

SOLUTION.

$x^3+3x=14$	(1).
$x^4+3x^2=14x$	$(2)=(1)\times x$. [members.
$x^4+7x^2=4x^2+14x$	$(3)=(2)$ with $4x^2$ added to both
$x^4+7x^2+(\frac{7}{2})^2=4x^2+14x+(\frac{7}{2})^2$.	$(4)=(3)$ square completed.
$x^2+\frac{7}{2}=2x+\frac{7}{2}$, or $-2x-\frac{7}{2}$,	$(5)=\sqrt{(4)}$.

$\therefore\ x=2$; or $x^2+2x=-7$

$x=-1\pm\sqrt{-6}$.

EXAMPLES.

1. Given $x^3-7x=-6$ to find x. *Ans.* $x=1$, 2, or -3.

2. Given $x^3-32x=-24$ to find x. *Ans.* $x=-6$, or $3\pm\sqrt{5}$.

3. Given $x^3-22x=24$ to find x. *Ans.* $x=-4$, or $2\pm\sqrt{10}$.

4. Given $x^3+6x=88$ to find x. *Ans.* $x=4$, or $-2\pm3\sqrt{-2}$.

5. Given $x^3+6x=45$ to find x. *Ans.* $x=3$, or $\dfrac{-3\pm\sqrt{-51}}{2}$.

6. Given $x^3-13x=-12$ to find x. *Ans.* $x=1$, 3, or -4.

7. Given $x^3+48x=104$ to find x. *Ans.* $x=2$, or $-1\pm\sqrt{-51}$.

8. Given $x^3-6x=9$ to find x. *Ans.* $x=3$, or $\dfrac{-3\pm\sqrt{-3}}{2}$.

9. Given $x+7x^{\frac{1}{3}}=22$ to find x. *Ans.* $x=8$, or $29\pm7\sqrt{-10}$.

10. Given $x^3+3x=14$ to find x. *Ans.* $x=2$, or $-1\pm\sqrt{-6}$.

11. Given $x^2-\dfrac{2}{3x}=1\frac{4}{9}$ to find x. *Ans.* $x=-\dfrac{2}{3}$, or $\dfrac{1\pm\sqrt{10}}{3}$.

12. Given $x^3-2x=-4$ to find x. *Ans.* $x=-2$, or $1\pm\sqrt{-1}$.

13. Given $5x^3+2x=44$ to find x. *Ans.* $x=2$, or $\dfrac{-5\pm\sqrt{-85}}{5}$.

14. Given $x^3+108x=665$ to find x. *Ans.* $x=5$, or $\dfrac{-5\pm13\sqrt{-3}}{2}$.

15. Given $x^3-39x=-70$ to find x. *Ans.* $x=2$, 5, or -7.

16. Given $x^3-49x=120$ to find x. *Ans.* $x=8$, -3, or -5.

17. Given $x^3+12x=-63$ to find x. *Ans.* $x=-3$, or $\dfrac{3\pm5\sqrt{-3}}{2}$.

18. Given $x^3-21x=-344$ to find x.

Ans. $x=-8$, or $4\pm3\sqrt{-3}$.

19. Given $x^3-6x=40$ to find x. *Ans.* $x=4$, or $-2\pm\sqrt{-6}$.

20. Given $x^3-7\frac{1}{2}x=-290\frac{1}{2}$ to find x.

Ans. $x=-7$, or $\dfrac{7\pm\sqrt{117}}{2}$.

(332.) Sometimes the factors of an equation are very apparent, and then the root may be obtained as in the solution to the following

PROBLEM.

Given $x^3-3x-2=0$ to find the value of x.

SOLUTION.

$$x^3-3x-2=0$$
$$x^3-x=2x+2$$
$$x(x^2-1)=2(x+1)$$

Since $x+1$ is a factor of both members of the equation, the equation will be satisfied by putting $\quad x+1=0$

Whence, $\quad x=-1.$

To get the other values of x divide both members by $x+1$.

And then, $\quad x^2-x=2$

$$x=2, \text{ or } -1.$$

EXAMPLES.

1. Given $x^3-5x=-4$ to find x. *Ans.* $x=1$, or $\dfrac{-1\pm\sqrt{17}}{2}$

2. Given $x^3-2x=1$ to find x. *Ans.* $x=-1$, or $\dfrac{1\pm\sqrt{5}}{2}$.

3. Given $x^3-2x=-1$ to find x. *Ans.* $x=1$, or $\dfrac{-1\pm\sqrt{5}}{2}$.

4. Given $x^3-8x=-8$ to find x. *Ans.* $x=2$, or $-1\pm\sqrt{5}$.

5. Given $x-1=2+\dfrac{2}{x^{\frac{1}{2}}}$ to find x. *Ans.* $x=1$, 1, or 4.

6. Given $\dfrac{x^2+3x-7}{x+2+\dfrac{18}{x}}=1$ to find x. *Ans.* $x=3$, -2, or -3.

7. Given $3x^3-7x^2=7x-3$ to find x. *Ans.* $x=\frac{1}{3}$, 3, or -1.

8. Given $x^3+x^2+x+1=10x+10$ to find x.
Ans. $x=3$, -1, -3.

9. Given $x^3-x^2-2x=-2$ to find x. *Ans.* $x=1$, or $\pm\sqrt{2}$.

10. Given $(x^2+2x)(x+4)=2-(x+4)$ to find x.
Ans. $x=-2$, or $-2\pm\sqrt{3}$.

11. Given $\left\{\begin{array}{l} 3x-\dfrac{3x}{y}=y^2-y, \\ y^2+x=4. \end{array}\right\}$ to find x and y.

Ans. $\left\{\begin{array}{l} x=3, \text{ or } 1. \\ y=1, \text{ or } \pm\sqrt{3}. \end{array}\right.$

(333.) It sometimes happens that the coefficients of the unknown quantity have such relations that the equation may be reduced according to the method adopted in the solution of the following

PROBLEM.

Given $x^3-6x^2+11x=6$ to find the values of x.

SOLUTION.

$$x^3-6x^2+11x-6=0$$
$$x^4-6x^3+11x^2-6x=0$$
$$x^4-6x^3+9x^2+2x^2-6x=0$$
$$(x^2-3x)^2+2(x^2-3x)=0 \qquad (a)$$
$$(x^2-3x)^2+2(x^2-3x)+1=1$$
$$x^2-3x+1=\pm 1$$
$$x^2-3x=0, \text{ or } -2,$$
$$\therefore\ x^2=3x; \qquad \text{or } x^2-3x=-2$$
$$x=3 \qquad\qquad x=\frac{3\pm 1}{2}=2, \text{ or } 1.$$

In equation (a) put $y=x^2-3x$, and we shall have

$$y^2+2y=0$$
$$y^2+2y+1=1$$
$$y+1=\pm 1$$
$$y=0, \text{ or } -2,$$
$$\therefore\ x^2-3x=0;\ \text{or } -2, \text{ the same as before.}$$

REMARK.—Judicious substitution often saves figures, and also often reveals relations which might not otherwise be so apparent.

Since x^2-3x is a factor of (a), we have immediately

$$x^2-3x=0, \text{ and by dividing by this,}$$
$$x^2-3x+2=0, \text{ the same as before.}$$

EXAMPLES.

1. Given $x^3-6x^2+5x+12=0$ to find x. *Ans.* $x=-1$, 3, or 4.

2. Given $x^3-2x^2-x+2=0$ to find x. *Ans.* $x=1$, 2, or -1.

3. Given $x^3-6x^2+12x-9=0$ to find x.

Ans. $x=3$, or $\frac{3\pm\sqrt{-3}}{2}$.

4. Given $x^3-2x^2-5x+6=0$ to find x. *Ans.* $x=1$, 3, or -2.

5. Given $x^3+2x^2-5x-6=0$ to find x. *Ans.* $x=2$, -1, or -3.

6. Given $x^3+6x^2+11x+6=0$ to find x.

Ans. $x=-1$, -2, or -3.

7. Given $x^3-8x^2+19x-12=0$ to find x. *Ans.* $x=1$, 3, or 4.

8. Given $x^3+2ax^2+5a^2x+4a^3=0$ to find x.

Ans. $x=-a$, or $\frac{-a\pm a\sqrt{-15}}{2}$.

(334.) The following examples may be solved according to the artifices employed in quadratics.

PROBLEM.

Given $x^3-3x^2+3x=9$ to find the values of x.

SOLUTION.

$$x^3-3x^2+3x=9 \qquad (1).$$
$$x^3-3x^2+3x-1=8 \qquad (2)=(1) \text{ with } -1 \text{ added to both members.}$$
$$x-1=2 \qquad (3)=\sqrt[3]{(2)}.$$
$$x=3$$

To obtain the other values find the other factor of x^3-3x^2+3x-9, $x-3$ being one. That factor is x^2+3; hence, we have

$$x^2+3=0.$$
$$x^2=-3.$$
$$x=\pm\sqrt{-3}.$$

The best method of solving this equation is by factoring, for its factors are very apparent.

$$x^3-3x^2+3x-9=0.$$
$$x^2(x-3)+3(x-3)=0.$$
$$\text{whence,} \qquad (x^2+3)(x-3)=0.$$
$$\therefore \qquad x-3=0.$$
$$\text{and} \qquad x^2+3=0.$$
$$\text{whence,} \qquad x=3, \text{ or } \pm\sqrt{-3}.$$

REMARK.—The examples which follow are not all of the character of the one whose solution has just been given, but embody different principles with which the student is supposed to be familiar.

EXAMPLES.

1. Given $x^3+6x^2+12x=19$ to find x.

Ans. $x=1$, or $\dfrac{-7\pm3\sqrt{-3}}{2}$.

2. Given $\sqrt{x^3+8}=\sqrt{125-6x^2-12x}$ to find x.

Ans. $x=3$, or $\dfrac{-9\pm5\sqrt{-3}}{2}$.

3. Given $\sqrt[3]{x^3-a^3}=\sqrt[3]{3ax^2-3a^2x+8b}$ to find x.

Ans. $x=a+2\sqrt[3]{b}$.

4. Given $\dfrac{\sqrt{a+x}}{a}+\dfrac{\sqrt{a+x}}{x}=\dfrac{\sqrt{x}}{c}$ to find x. *Ans.* $x=\dfrac{ac^{\frac{2}{3}}}{a^{\frac{2}{3}}-c^{\frac{2}{3}}}$.

5. Given $x^3-6x^2+12x=8$ to find x. *Ans.* $x=2, 2, 2$.

6. Given $x^3-15x^2+75x=125$ to find x. *Ans.* $x=5, 5, 5$.

HENKLE'S METHOD.*

(335.) The following tentative process for solving affected and incomplete cubics, has never before been given in a work upon algebra.

PROBLEM.

1. Given $x^3-8x^2+11x+20=0$ to find the values of x.

SOLUTION.

$$x^3-8x^2+11x+20=0.$$
$$x^4-8x^3+11x^2+20x=0.$$
$$(x^2-4x)^2-5x^2+20x=0. \qquad (A).$$
$$(x^2-4x)^2-4(x^2-4x)+4=x^2-4x+4. \qquad (B).$$
$$x^2-4x-2=x-2, \text{ or } 2-x,$$
$$\therefore \quad x^2=5x; \text{ or } x^2-3x=4.$$
$$x=5 \qquad x=\frac{3\pm5}{2}=4, \text{ or} -1.$$

This solution will need some explanation. Every affected cubic equation may be put into the form expressed by (A) that is, made to consist of the square of a binomial followed by two terms. If these two terms contain as a factor the first power of the binomial, the equation may be solved as in (B); but if not, we may proceed as in the above example. Let us resume the equation

$$(x^2-4x)^2-5x^2+20x=0.$$

Let us see whether x^2 is not the first term of a binomial square, which being added to both members of this equation will render them perfect squares. If x^2 is added to both members, we see that the coefficient of the second term in (B) is -4; considering $(x^2-4x)^2$ as the first term; and this coefficient shows that the third term in the first member of (B) must be 4, and therefore, the third term in the second member must also be 4; but if the first and the last term of a binomial square are x^2 and 4, the middle term must be either $+4x$, or $-4x$. From which we see that the square of the binomial of which x^2 is the first term must, in this case, be $x^2\pm4x+4$. Since, 4 is in both members, it follows that if $x^2\pm4x$ be added to $-5x^2+20x$, and the result should be $-4(x^2-4x)$, or $-4x^2+16x$, that $x^2\pm4x+4$ is the proper square to add. As we have already added x^2, it only remains for us to see whether $4x$ added to or subtracted from $+20x$

* This method of solving Cubic Equations, which is also applied to the solution of Biquadratics was discovered by Prof. W. D. Henkle. I, therefore, have assumed the responsibility of calling it the "Henkle Method."—J. F. S.

makes $+16x$. It will be seen that $4x$ subtracted, or what is the same $-4x$ added makes $+20x$ become $+16x$, therefore, x^2-4x+4 is the proper square to add.

The student in passing from (A) to (B) should put his trial work one side.

PROBLEM.

2. Given $x^3+2x^2-33x+14=0$.

SOLUTION.

$$x^3+2x^2-33x+14=0,$$
$$x^4+2x^3-33x^2+14x=0,$$
$$(x^2+x)^2-34x^2+14x=0, \quad (A).$$
$$(x^2+x)^2+2(x^2+x)+1=36x^2-12x+1. \quad (B.)$$
$$x^2+x+1=6x-1, \text{ or } 1-6x.$$
$$\therefore\ x^2-5x=-2; \ . \text{ or } x^2=-7x,$$
$$x=\frac{5\pm\sqrt{17}}{2} \qquad x=-7.$$

Trial work between (A) and (B).

$$-33(x^2+x)+(\tfrac{33}{2})^2 \qquad x^2\pm33x+(\tfrac{33}{2})^2.$$
$$-30(x^2+x)+(15)^2 \qquad 4x^2\pm30x+(15)^2.$$
$$-25(x^2+x)+(\tfrac{25}{2})^2 \qquad 9x^2\pm75x+(\tfrac{25}{2})^2.$$
$$-18(x^2+x)+81 \qquad 16x^2\pm72x+81.$$
$$-\ 9(x^2+x)+\ (\tfrac{9}{2})^2 \qquad 25x^2\pm45x+(\tfrac{9}{2})^2.$$
$$+\ 2(x^2+x)+1 \qquad 36x^2\pm12x+1.$$

The work upon the right shows the trial work. First x^2 was added, which resulted in $\pm33x$ for the middle term for the second member, but since $+33x$ or $-33x$ added to $+14x$, does not give $-33x$, we see that x^2 is not the proper term to add in the start. So when $4x^2$

is added, $+60x$ or $-60x$ added to $+14x$ does not give $-30x$; or when $9x^2$ is added, $+75x$, or $-75x$ added to $+14x$ does not give $-25x$; or when $16x^2$ is added, $+72x$, or $-72x$ added to $+14x$ does not give $-18x$; or when $25x^2$ is added, $+45x$, or $-45x$ added to $+14x$ does not give $-9x$; but when $36x^2$ is added, $-12x$ added to $+14x$ *does* give $+2x$.

The failure in adding x^2 is so much that the operator would not likely try the successive squares, but would immediately pass to $16x^2$, $25x^2$ or $36x^2$.

REMARK.—The above is given to show the mode that the student should pursue in finding the proper square. One who is skillful in this mode of solution seldom makes more than three trials, and frequently but one.

If the first term of the equation is x^3 and the coefficient of the second term is odd, multiply each term by $4x$.

If the first term is not a square, it should be made one by multiplying the equation by four times the coefficient of the first term. When the coefficient of the second is even, it is frequently only necessary to multiply by once the coefficient of the first term. It is not absolutely necessary to follow the last directions given, but if they are followed, the student will be saved the trouble of operating with fractions which he might otherwise encounter.

PROBLEM

3. Given $x^3+5x^2+3x-9=0$ to find the value of x.

SOLUTION.

$$x^3+5x^2+3x-9=0.$$
$$4x^4+20x^3+12x^2-36x=0.$$
$$(2x^2+5x)^2-13x-36x=0.$$
$$(2x^2+5x)^2-6(2x^2+5x)+9=x^2+6x+9.$$
$$2x^2+5x-3=x+3, \text{ or } -x-3,$$
$$\therefore \quad x^2+2x=3; \qquad \text{or } 2x^2=-6x,$$
$$x=-1\pm2=1, \text{ or } -3 \qquad x=-3.$$

PROBLEM

4. Given $3x^3-14x^2+21x-10=0$ to find the value of x.

SOLUTION.

$$3x^3-14x^2+21x-10=0.$$
$$9x^4-42x^3+63x^2-30x=0.$$
$$(3x^2-7x)^2+14x^2-30x=0.$$
$$(3x^2-7x)^2+5(3x^2-7x)+\overline{\tfrac{5}{2}}|^2=x^2-5x+\overline{\tfrac{5}{2}}|^2.$$
$$3x^2-7x+\tfrac{5}{2}=x-\tfrac{5}{2}, \text{ or } \tfrac{5}{2}-x,$$
$$3x^2-8x=-5; \qquad \text{or } 3x^2=6x,$$
$$x=\frac{4\pm1}{3}=\tfrac{5}{3} \text{ or } 1 \qquad x=2.$$

(336.) Since the solving of cubics by this process is a good mental exercise, a copious list of examples is appended. The pupil before commencing should make himself familiar with the perfect squares from 1 to 1296.

EXAMPLES.

1. Given $x^3-6x^2+13x-10=0$ to find x.

Ans. $x=2$, or $2\pm\sqrt{-1}$.

2. Given $x^3+6x^2-7x-60=0$ to find x. *Ans.* $x=3, -4$, or -5.

3. Given $4x^3-24x^2+45x-25=0$ to find x.

Ans. $x=1, 2\frac{1}{2}$, or $2\frac{1}{2}$.

4. Given $x^3-9x^2-130x+600=0$ to find x.

Ans. $x=4, 15$, or -10.

5. Given $x^3+6x^2-32=0$ to find x. *Ans.* $x=2, -4$, or -4.

6. Given $x^3-11x^2+43x-65=0$ to find x.

Ans. $x=5$, or $3\pm\sqrt{-4}$.

7. Given $x^3-23x^2+167x-385=0$ to find x.

Ans. $x=5, 7$, or 11.

8. Given $x^3-12x^2+36x-7=0$ to find x.

Ans. $x=7$, or $\dfrac{5\pm\sqrt{21}}{2}$.

9. Given $x^3+8x^2+17x+10=0$ to find x.

Ans. $x=-1, -2$, or -5.

10. Given $x^3+6x^2+20x+15=0$ to find x.

Ans. $x=-1$, or $\dfrac{-5\pm\sqrt{-35}}{2}$.

11. Given $24x^3-26x^2+9x-1=0$ to find x. *Ans.* $x=\frac{1}{4}, \frac{1}{3}$, or $\frac{1}{2}$.

12. Given $x^3-24x^2+191x-504=0$ to find x.

Ans. $x=7, 8$, or 9.

13. Given $x^3-13x^2+49x-45=0$ to find x.

Ans. $x=5$, or $4\pm\sqrt{7}$.

14. Given $2x^3-3x^2+2x-3=0$ to find x.

Ans. $x=1\frac{1}{2}$, or $\pm\sqrt{-1}$.

15. Given $2x^3-15x^2+25x+6=0$ to find x.

Ans. $x=3$, or $\dfrac{9\pm\sqrt{97}}{4}$.

16. Given $2x^3-12x^2+13x-15=0$ to find x.

Ans. $x=5$, or $\frac{1\pm\sqrt{-5}}{2}$.

17. Given $x^3+10x^2-101x-990=0$ to find x.

Ans. $x=10$, -9, or -11.

18. Given $x^3-2x^2-33x+90=0$ to find x.

Ans. $x=3$, 5, or -6.

19. Given $4x^3-24x^2+21x-5=0$ to find x. *Ans.* $x=\frac{1}{2}$, $\frac{1}{2}$, or 5.

20. Given $2x^3-33x^2+121x+84=0$ to find x.

Ans. $x=7$, or $\frac{19\pm\sqrt{457}}{4}$.

21. Given $x^3+x^2-34x+56=0$ to find x. *Ans.* $x=2$, 4, or -7.

22. Given $x^3-21x^2+146x-336=0$ to find x.

Ans. $x=6$, 7, or 8.

23. Given $x^3-\cdot3x^2+\cdot004=0$ to find x. *Ans.* $x=\cdot2$, $\cdot2$, or $-\cdot1$.

24. Given $x^3-10x^2+10x-100=0$ to find x.

Ans. $x=10$, or $\pm\sqrt{-10}$.

25. Given $x^3-2x^2+3x-4\frac{2}{27}=0$ to find x.

Ans. $x=1\frac{2}{3}$, or $\frac{1\pm\sqrt{-87}}{6}$.

26. Given $x^3-4x^2-28x-32=0$ to find x.

Ans. $x=8$, -2, or -2.

27. Given $2x^3-15x^2+25x-6=0$ to find x.

Ans. $x=2$, or $\frac{11\pm\sqrt{97}}{4}$.

28. Given $x^3-x^2-7x+15=0$ to find x.

Ans. $x=-3$, or $2\pm\sqrt{-1}$.

29. Given $2x^3-6x^2+4x-120=0$ to find x.

Ans. $x=5$, or $-1\pm\sqrt{-11}$.

30. Given $x^3+9x^2-82x-720=0$ to find x.

Ans. $x=9$, -8, or -10.

31. Given $x^3-x^2-10x+6=0$ to find x.

Ans. $x=-3$, or $2\pm\sqrt{2}$.

32. Given $8x^3-26x^2+11x+10=0$ to find x.

Ans. $x=2\frac{1}{2}$, or $\frac{3\pm\sqrt{41}}{8}$.

33. Given $2x^3+12x^2+48x-790=0$ to find x.

Ans. $x=5$, or $\dfrac{-11\pm\sqrt{-195}}{2}$.

34. Given $2x^3-3x^2+x-30=0$ to find x.

Ans. $x=3$, or $\dfrac{-3\pm\sqrt{-71}}{4}$,

35. Given $x^3+17x^2-140x-2496=0$ to find x.

Ans. $x=12$, -13, or -16.

36. Given $x^3+3x^2-54=0$ to find x. *Ans.* $x=3$, or $-3\pm\sqrt{-9}$.

37. Given $x^3+x^2-17x+15=0$ to find x. *Ans.* $x=1$, 3, or -5.

38. Given $4x^3-48x^2+45x-11=0$ to find x.

Ans. $x=\frac{1}{2}$, $\frac{1}{2}$, or 11.

39. Given $2x^3+3x^2+x-30=0$ to find x.

Ans. $x=2$, or $\dfrac{-7\pm\sqrt{-71}}{4}$.

40. Given $6x^3+7x^2+39x+63=0$ to find x.

Ans. $x=-1\frac{1}{2}$, or $\frac{1}{6}\pm\frac{1}{6}\sqrt{-251}$.

41. Given $x^3+6x^2-1600=0$ to find x.

Ans. $x=10$, $-8\pm4\sqrt{-6}$.

42. Given $2x^3-33x^2+121x-84=0$ to find x.

Ans. $x=4$, or $\dfrac{25\pm\sqrt{457}}{4}$.

43. Given $x^3+6x^2-3920=0$ to find x.

Ans. $x=14$, or $-10\pm6\sqrt{-5}$.

44. Given $2x^3-9x^2+9x-308=0$ to find x.

Ans. $x=7$, or $\dfrac{-5\pm\sqrt{-327}}{4}$.

45. Given $x^3-29x^2+198x-360=0$ to find x.

Ans. $x=3$, 6, or 20.

46. Given $2x^3+9x^2+9x-308=0$ to find x.

Ans. $x=4$, or $\dfrac{-17\pm\sqrt{-327}}{4}$.

47. Given $4x^3-112x^2+109x-27=0$ to find x.

Ans. $x=\frac{1}{2}$, $\frac{1}{2}$, or 27.

48. Given $8x^3+34x^2-79x+30=0$ to find x.

Ans. $x=\frac{1}{2}$, $1\frac{1}{4}$, or -6.

49. Given $4x^3-152x^2+149x-37=0$ to find x.

Ans. $x=\frac{1}{2}$, $\frac{1}{2}$, or 37.

50. Given $x^3-15x^2+74x-120=0$ to find x. *Ans.* 4, 5, or 6.

(337.) The solution of the following literal equation will show that there always is a binomial square, which being added to both members of a cubic equation, after it has been multiplied by x, that will render both sides perfect squares. The only difficulty consists in finding the binomial square.

It will be perceived by looking at the values of the unknown quantity in the preceding examples, that this mode of solution is practicable when one or more of the values of the unknown quantity is a whole number, and also, when one or more of the values is a mixed number, or a proper fraction.

PROBLEM.

Given $x^3-(a+b+c)x^2+(ab+ac+bc)x-abc=0$ to find the values of x.

SOLUTION.

$$4x^4-4(a+b+c)x^3+4(ab+ac+bc)x^2-4abc=0,$$
$$[2x^2-(a+b+c)x]^2-(a^2-2ab-2ac+b^2-2bc+c^2)x^2-4abc=0,$$
$$[2x^2-(a+b+c)x]^2+2bc[2x^2-(a+b+c)x]+b^2c^2=(a-b-c)^2x^2+2bc(a-b-c)x+b^2c^2,$$
$$2x^2-(a+b+c)x+bc=(a-b-c)x+bc, \text{ or } -(a-b-c)x-bc,$$
$$\therefore\ 2x=2a; \text{ or } 2x^2-2(b+c)x=-2bc,$$
$$x=a \qquad x=\frac{b+c\pm(b-c)}{2}=b, \text{ or } c.$$

REMARK.—An inspection of this example will enable the student to see what relation exists between the roots of the equation and the added binomial square.

QUESTIONS.

1. What number is it whose third part multiplied by its square gives 1944? *Ans.* 18.

2. What number is it whose $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$, multiplied together, and the product increased by 32, gives 4640? *Ans.* 48.

3. There is a number such, that its 4th power divided by the 8th part of the number, and 167 subtracted from the quotient, the remainder is 12000. What number is it? *Ans.* $11\frac{1}{2}$.

4. Some merchants engage in business; each gives to it 1000 times as many dollars as there are partners. They gain in the business \$2560; and it is found that they have gained exactly half their own number per cent. How many merchants are there? *Ans.* 8.

5. A capitalist puts out \$10000 at interest, and adds the interest yearly to the capital. At the end of the third year he finds his capital increased to \$11576$\frac{1}{4}$. How much per cent. did he receive?

Ans. 5 per cent.

6. There are two numbers whose difference is 4, and, moreover, are such that their product multiplied by their sum gives 1386. What numbers are they? *Ans.* 7 and 11.

7. Some officers were in a field with a detachment, partly infantry and partly cavalry. Each officer had under his command 3 times as many cavalry and 7 times as many foot as there are officers. Each cavalry soldier has 2, and each foot soldier 22 cartridges more than there are officers. They had altogether 15360 cartridges. How many officers were there? *Ans.* 8.

8. A person being asked how much he had expended that day, answered, "I have spent \$4 more, and yesterday twice as much as I did the day before yesterday; if I multiply together the sums which I expended in dollars during these three days, and add 756 to the product, I obtain exactly 134 times as much as I have expended to-day." How much did he expend that day? *Ans.* 6 or 9 dollars.

9. Some merchants jointly form a certain capital, in such a way that each contributes 10 times as many dollars as they are in number; they trade with this capital, and gain as many dollars per cent. as exceed the number of merchants by 8. Their profit amounts to \$288. How many were there of them? *Ans.* 12.

10. Some merchants collect a capital of \$8240. To this each contributes 40 times as many dollars as there are of them. With this whole sum they gain as many dollars per cent. as there are persons. They then divide the profit; and each takes 10 times as many dollars as there are persons; but after this there remains \$224. How many merchants were there? *Ans.* Either 7, 8, or 10.

11. The 3d power of a number added to 9 times its 2d power, 27 times the number and 27 more, is equal to 125. What is the number? *Ans.* 2.

12. If the difference of two numbers be multiplied by the 2d power of the greater, and the sum of the two numbers by the 2d power of the greater, the sum of the two products will be 432; and the difference of the products 280. What are the numbers?

Ans. 6 and 3$\frac{2}{3}$.

13. A and B commenced to speculate, each with the same sum of money; after a certain number of months, A had the 3d power of the number of dollars which he had at first, wanting 36 times its 2d power. B had 432 times as much as he had at first, wanting \$1728; then the sum of what A and B had was equal to \$343. What sum had A and B at first? *Ans.* \$19.

14. There are three numbers; the second is 2, and the third 3 more than the first, and their continued product is 40. What are the numbers? *Ans.* 2, 4, and 5.

15. A has \$4 more than B; but, if the number A has be multiplied by the 2d power of the number B has, the product will be 225. What number of dollars has each? *Ans.* A \$9 and B \$5.

16. * The sum of two numbers is 10, and the difference of their 4th powers is 1040. What are the numbers? *Ans.* 6 and 4.

17. The difference of 2 numbers is 8, and the difference of their 4th powers is 14560. What are the numbers? *Ans.* 11 and 3.

18. The joint capital of 3 partners was \$65000. A's money was in trade 3 months, B's 4 months, and C's 5 months. When they shared stock and gain A received \$3900, B received \$2700, and C's gain was \$750. What was each partner's share of the stock?

Ans.

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* NOTE—Let $x+y$ and $x-y$ represent the numbers. This mode of representation frequently produces equations simpler than those produced by the usual representation.

CHAPTER XIII.

BIQUADRATIC EQUATIONS.

(338.) Biquadratic Equations may be divided into three classes; namely,

Pure Biquadratics;
Incomplete Biquadratics; and
Affected Biquadratics.

339.) A *Pure Biquadratic* is an equation in which the unknown quantity is contained in but one term, and its exponent is 4, or a fraction whose numerator is 4 and whose denominator is an odd number; as,

$$ax^4 = \pm b;\ cx^{\frac{4}{3}} = \pm b;\ bx^{\frac{4}{5}} = \pm c,\ \&c.$$

PROBLEM.

(340.) Given $x^4=1$ to find the value of x.

SOLUTION 1.

$$x^4=1,$$
$$x^2=\pm 1,$$
$$x=\pm\sqrt{\pm 1},$$
$$\therefore\quad x=1,\ -1,\ \sqrt{-1},\ \text{or}\ -\sqrt{-1}.$$

SOLUTION 2.

$$x^4=1,$$
$$x^4-1=0,$$
$$(x^2-1)(x^2+1)=0,$$
$$\text{whence,}\quad x^2-1=0,$$
$$\text{and}\quad x^2+1=0,$$
$$\therefore\quad x^2=1,$$
$$\text{and}\quad x^2=-1,$$
$$\text{whence,}\quad x=1,\ \text{or}\ -1,$$
$$\text{and}\quad x=\sqrt{-1},\ \text{or}\ -\sqrt{-1}.$$

SOLUTION 3.

$$x^4=1,$$
$$x^4-1=0,$$
$$(x-1)(x^3+x^2+x+1)=0,$$
$$\text{whence,}\quad x-1=0,$$
$$\text{and}\quad x^3+x^2+x+1=0 \qquad (a)$$
$$x=1.$$

Equation (a) is a cubic, and is solved thus,

$$4x^4+4x^3+4x^2+4x=0 \qquad (b)=(a)\times 4x.$$
$$(2x^2+x)^2+3x^2+4x=0,$$
$$(2x^2+x)^2+2(2x^2+x)+1=x^2-2x+1,$$
$$2x^2+x+1=x-1, \text{ or } 1-x,$$
$$\therefore\quad 2x^2=-2\,; \text{ or } 2x^2=-2x,$$
$$x^2=-1 \qquad 2x=-2,$$
$$x=\pm\sqrt{-1} \qquad x=-1.$$

SOLUTION 4.

$$x^3+x^2+x+1=0,$$
$$x^2(x+1)+(x+1)=0,$$
$$(x+1)(x^2+1)=0,$$
$$\text{whence,}\quad x+1=0,$$
$$\text{and}\quad x^2+1=0,$$
$$\therefore\quad x=-1,$$
$$\text{and}\quad x^2=-1,$$
$$x=\pm\sqrt{-1}.$$

EXAMPLES.

1. Given $x^4=16$ to find x. *Ans.* $x=\pm 2, \pm 2\sqrt{-1}$.

2. Given $3x^4=243$ to find x. *Ans.* $x=\pm 3, \pm 3\sqrt{-1}$.

3. Given $x^4=256$ to find x. *Ans.* $x=\pm 4, \pm 4\sqrt{-1}$.

4. Given $x^4=a^4$ to find x. *Ans.* $x=\pm a, \pm a\sqrt{-1}$.

5. Given $x^4=b$ to find x. *Ans.* $x=\pm\sqrt{\pm b}$.

6. Given $ax^4=c$ to find x. *Ans.* $x=\pm\sqrt{\pm\frac{c}{a}}$.

AFFECTED AND INCOMPLETE BIQUADRATICS.

(341.) An *Affected Biquadratic* is an equation in which the unknown quantity occurs in but four terms, the least exponent of the unknown quantity being 1, or a proper fraction whose numerator is 1, and the exponents of the unknown quantity in the other terms being respectively two, three, and four times as large; as,

$$x^4+ax^3+bx^2+cx=d;\ x^2+ax^{\frac{3}{2}}+bx+cx^{\frac{1}{2}}=d, \text{ \&c.}$$

(342.) An affected biquadratic becomes an *incomplete biquadratic* by omitting any one or two of the terms which contain the unknown quantity, after the first term, considering the equation to be arranged so that the exponents form a descending series, and the first term to be that which has the greatest exponent, and provided that when the exponents of the unknown quantity in the second and fourth terms are fractions whose denominators are even numbers, these terms shall not be omitted. Thus,

$$x^4+ax^3+bx^2=d;\ x^4+ax^3+cx=d;\ x^4+bx^2+cx=d;\ x^4+ax^3=d;$$
$$x^4+cx=d;\ x^4+bx^2=d.$$

$$x^2+ax^{\frac{3}{2}}+bx=d;\ x^2+ax^{\frac{3}{2}}+cx^{\frac{1}{2}}=d;\ x^2+bx+cx^{\frac{1}{2}}=d;\ x^2+ax^{\frac{3}{2}}=d;$$
$$x^2+cx^{\frac{1}{2}}=d;\ x^{\frac{4}{3}}+bx^{\frac{2}{3}}=d,$$

are incomplete quadratics.

(343.) An incomplete equation of the form $ax^4 \pm bx^2 = \pm c$, or $ax^{\frac{4}{n}} \pm bx^{\frac{2}{n}} = \pm c$, n being an odd nnmber, is susceptible of a general solution.

PROBLEM.

Given $ax^4+bx^2=c$ to find the values of x.

SOLUTION.

$$ax^4+bx^2=c,$$
$$4a^2x^4+4abx^2=4ac,$$
$$4a^2x^4+4abx^2+b^2=b^2+4ac,$$
$$2ax^2+b=\pm\sqrt{b^2+4ac}.$$
$$2ax^2=-b\pm\sqrt{b^2+4ac},$$
$$x^2=\frac{-b\pm\sqrt{b^2+4ac}}{2a},$$
$$x=\pm\sqrt{\frac{-b\pm\sqrt{b^2+4ac}}{2a}}.$$

(344.) It is sometimes advisable to consider two or more terms as one.

PROBLEM.

Given $9x-4x^2+\sqrt{4x^2-9x+11}=5$ to find the values of x.

SOLUTION.

$$9x-4x^2+\sqrt{4x^2-9x+11}=5,$$
$$4x^2-9x-\sqrt{4x^2-9x+11}=-5,$$
$$4x^2-9x+11-\sqrt{4x^2-9x+11}=6.$$

Considering $\sqrt{4x^2-9x+11}$ as one term, and putting it equal to y, the equation becomes

$$y^2-y=6,$$
$$y=\frac{1\pm5}{2}=3, \text{ or } -2,$$
$$\therefore\ 4x^2-9x+11=y^2=9, \text{ or } 4,$$
$$4x^2-9x=-2, \text{ or } -7,$$
$$x=\frac{9\pm7}{8}, \text{ or } \frac{9\pm\sqrt{-31}}{8},$$
$$x=2, \text{ or } \tfrac{1}{4};\ \text{or } \frac{9\pm\sqrt{-31}}{8}.$$

EXAMPLES.

1. Given $x^4+4x^2=117$ to find x. *Ans.* $x=3$.

2. Given $x^{\frac{4}{3}}+7x^{\frac{2}{3}}=44$ to find x. *Ans.* $x=\pm8$, or $(-11)^{\frac{3}{2}}$.

3. Given $x^4-74x^2=-1225$ to find x. *Ans.* $x=\pm5$, or ±7.

4. Given $x^4-6x^2=27$ to find x. *Ans.* $x=\pm3$, or $\pm\sqrt{-3}$.

5. Given $x^2+11+\sqrt{x^2+11}=42$ to find x.

Ans. $x=\pm5$, or $\pm\sqrt{38}$.

6. Given $x^2-7x+\sqrt{x^2-7x+18}=24$ to find x.

Ans. $x=9$, -2, or $\frac{7\pm\sqrt{173}}{2}$.

7. Given $x^2+\sqrt{5x+x^2}=42-5x$ to find x.

Ans. $x=4$, -9, or $\frac{-5\pm\sqrt{221}}{2}$.

8. Given $(x^2+5)^2-4x^2=160$ to find x.

Ans. $x=\pm3$, or $\pm\sqrt{-15}$.

9. Given $\frac{1}{(2x-4)^2}=\frac{1}{8}+\frac{2}{(2x-4)^4}$ to find x. *Ans.* $x=3, 3, 1,$ or 1.

10. Given $x^2-2x+6\sqrt{x^2-2x+5}=11$ to find x.

Ans. $x=1$, 1 or $1\pm2\sqrt{15}$.

11. Given $2x^2+3x-5\sqrt{2x^2+3x+9}=-3$ to find x.

Ans. $x=3$, $-4\frac{1}{2}$, or $\frac{-3\pm\sqrt{-55}}{4}$.

12. Given $9x+\sqrt{16x^2+36x^3}=15x^2-4$ to find x.

Ans. $x=1\frac{1}{3}$, $-\frac{1}{3}$, or $\frac{9\pm\sqrt{481}}{50}$.

13. Given $\left(x+\frac{8}{x}\right)+x=42-\frac{8}{x}$ to find x.

Ans. $x=2$, 4, or $\frac{-7\pm\sqrt{17}}{2}$.

14. Given $x(\sqrt{x}+1)^2=102(x+\sqrt{x})-2576$ to find x.

Ans. $x=49$, 64, or $\frac{93\mp\sqrt{185}}{2}$.

15. Given $6x^2+2\sqrt{9x^2-24x}=16x+12$ to find x.

Ans. $x=3$, $-\frac{1}{3}$, or $\frac{4\pm2\sqrt{13}}{3}$.

16. Given $\frac{\sqrt{x^2+x+6}}{3}=\frac{18-(\frac{4}{3}\sqrt{x^2+x+6}-2)}{\sqrt{x^2+x+6}}$ to find x.

Ans. $x=5$, -6, or $\frac{-1\pm\sqrt{377}}{2}$.

17. Given $(x^2-1)(x^2-2)+(x^2-3)(x^2-4)=x^4+5$ to find x.

Ans. $x=\pm1$, or ±3.

18. Given $\frac{1}{x^2+11x-8}+\frac{1}{x^2+2x-8}+\frac{1}{x^2-13x-8}=0$ to find x.

Ans. $x=\pm1$, or ±8.

The resulting equation in this problem after reduction is

$$x^4-65x^2=-64.$$

In solving this, to avoid large numbers put $65=2a$, then $-64=-2a+1$. The equation after substituting becomes:

$$x^4-2ax^2=-2a+1,$$
$$x^4-2ax^2+a^2=a^2-2a+1,$$
$$x^2-a=a-1, \text{ or } 1, -a.$$
$$\therefore \quad x^2=2a-1; \quad \text{or } x^2=1,$$
$$x^2=64 \qquad x=\pm1,$$
$$x=\pm8.$$

(345.) Biquadratic equations may be reduced to simpler forms when both sides are perfect powers, zero being considered a perfect power, by extracting the root. Sometimes artifice is necessary to get the equation in a proper form for reducing. The artifices that may be employed are numerous; particular ones being applicable only to particular problems.

PROBLEM

1. Given $x^4-4x^3+6x^2-4x+1=0$ to find the values of x.

SOLUTION 1.

$$x^4-4x^3+6x^2-4x+1=0.$$

Taking square root twice, or the 4th root, we have

$$x-1=\pm\sqrt{\pm 0}=0,\ -0,\ \sqrt{-0},\ \text{or}\ -\sqrt{-0}=0,\ 0,\ 0,\ \text{or}\ 0.$$

$$x=1,\ 1,\ 1,\ \text{or}\ 1.$$

SOLUTION 2.

$$x^4-4x^3+6x^2-4x+1=0,$$

$$(x^2-1)(x-1)(x-1)(x-1)=0,$$

whence, $x-1=0,$

$x-1=0,$

$x-1=0,$

$x-1=0,$

$\therefore\quad x=1,\ 1,\ 1,\ \text{or}\ 1.$

PROBLEM

2. Given $x^4-4x^3+6x^2-4x^2+1=6$ to find the values of x.

SOLUTION.

$$x^4-4x^3+6x^2-4x+1=6.$$

$$x-1=\pm\sqrt{\pm 6},$$

$$x=1\pm\sqrt{\pm 6},$$

$$x=1\pm\sqrt{6},\ \text{or}\ 1\pm\sqrt{-6}.$$

Or,

$$x^2-2x+1=\pm\sqrt{6},$$

$$x-1=\pm\sqrt{\pm 6},$$

$$x=1\pm\sqrt{\pm 6}.$$

PROBLEM

3. Given $\frac{49x^2}{4}+\frac{48}{x^2}-49=9+\frac{6}{x}$ to find the values of x.

SOLUTION.

$$\frac{49x^2}{4}+\frac{48}{x^2}-49=9+\frac{6}{x}$$

$$\frac{49x^2}{4}-49+\frac{48}{x^2}=9+\frac{6}{x},$$

$$\frac{49x^2}{4}-49+\frac{49}{x^2}=9+\frac{6}{x}+\frac{1}{x},$$

$$\frac{7x}{2}-\frac{7}{x}=3+\frac{1}{x}, \text{ or } -3-\frac{1}{x},$$

$$\therefore \quad \frac{7x}{2}=3+\frac{8}{x}; \qquad \text{or } \frac{7x}{2}=-3+\frac{6}{x},$$

$$7x^2=6x+16 \qquad 7x^2=-6x+12,$$

$$7x^2-6x=16 \qquad 7x^2+6x=12,$$

$$x=\frac{3\pm 11}{7}=2, \text{ or } -1\tfrac{1}{7} \qquad x=\frac{-3\pm\sqrt{93}}{7}.$$

PROBLEM

4. Given $\left((x-2)^2-x\right)^2-(x-2)^2=88-(x-2)$ to find x.

SOLUTION.

$$\left((x-2)^2-x\right)^2-(x-2)^2=88-(x-2,)$$

$$\left((x-2)^2-x\right)^2-\left((x-2)^2-x\right)=90,$$

$$(x-2)^2-x=\frac{1\pm 19}{2}=10, \text{ or } -9,$$

$$\therefore \quad x^2-5x=6 \qquad ; \text{ or, } x^2-5x=-13,$$

$$x=\frac{5\pm 7}{2}=6, \text{ or } -1 \qquad x=\frac{5\pm 3\sqrt{-3}}{2}$$

PROBLEM

5. Given $x=\dfrac{12+8x^{\frac{1}{2}}}{x-5}$ to find x.

SOLUTION.

$$x=\frac{12+8x^{\frac{1}{2}}}{x-5},$$

$$x^2-5x=12+8x^{\frac{1}{2}},$$

$$x^2-4x=12+8x^{\frac{1}{2}}+x,$$

$$x^2-4x+4=16+8x^{\frac{1}{2}}+x.$$

$$x-2=4+x^{\frac{1}{2}}, \text{ or } -4-x^{\frac{1}{2}},$$

[forward.

$\therefore \quad x-x^{\frac{1}{2}}=6$; $\qquad$ or $x+x^{\frac{1}{2}}=-2$

$$x^{\frac{1}{2}}=\frac{1\pm5}{2}=3, \text{ or } -2 \qquad x^{\frac{1}{2}}=\frac{-1\pm\sqrt{-7}}{2}$$

$$x=9, \text{ or } 4 \qquad x=\frac{-3\mp\sqrt{-7}}{2}$$

EXAMPLES.

1. Given $x^4-8x^3+24x^2-32x+16=81$ to find x.

Ans. $x=5$, -1, or $2\pm3\sqrt{-1}$.

2. Given $x^4-8x^3+24x^2-32x=240$ to find x.

Ans. $x=6$, -2, or $2\pm4\sqrt{-1}$.

3. Given $\frac{x^4}{2}+\frac{17x^3}{4}-17x=8$ to find x.

Ans. $x=\pm2$, or $-\frac{1}{2}$, or -8.

4. Given $27x^2-\frac{841}{3x^2}+\frac{17}{3}=\frac{232}{3x}-\frac{1}{3x^2}+5$ to find x.

Ans. $x=2$, $-1\frac{5}{9}$, or $\frac{-2\pm\sqrt{-266}}{9}$.

5. Given $x^2-\frac{5x}{2}+15=\frac{25x^2}{16}-\frac{64}{x^2}$ to find x.

Ans. $x=4$, -8, or $\frac{-2\pm2\sqrt{-71}}{9}$.

6. Given $\frac{x^2}{(x^2-4)}+\frac{6}{x^2-4}=\frac{351}{25x^2}$ to find x.

Ans. $x=\pm3$, or $\pm\sqrt{\frac{32}{11}}$.

7. Given $x+4-2\sqrt{\frac{x+4}{x-4}}=\frac{3}{x-4}$ to find x.

Ans. $x=\pm5$, or $\pm\sqrt{17}$.

8. Given $3\left((x-1)^2-x\right)^2+2x=341+2(x-1)^2$ to find x.

Ans. $x=5$, -2, or $\frac{3\sqrt{3}\pm\sqrt{-109}}{2\sqrt{3}}$.

9. Given $\frac{x}{x+4}+\frac{4}{\sqrt{x+4}}=\frac{21}{x}$ to find x.

Ans. $x=12$, -3, or $\frac{49\pm\sqrt{3185}}{2}$.

10. Given $x^2-2=2\sqrt{5-4x}$ to find x.

Ans. $x=-1\pm\sqrt{5}$, or $1\pm\sqrt{-3}$.

11. Given $\dfrac{x+\sqrt{x^2-9}}{x-\sqrt{x^2-9}}=(x-2)^2$ to find x.

Ans. $x=3$, 5, or $\dfrac{8\pm\sqrt{-11}}{5}$.

12. Given $\dfrac{33\frac{24}{25}}{\sqrt{5x^2-x^4}}+\dfrac{\sqrt{5-x^2}}{25x}=\dfrac{34}{x}$ to find x.

Ans. $x=\pm 2$, or $\pm\sqrt{-720796}$.

13. Given $(x+6)^2+2x^{\frac{1}{2}}(x+6)=138+x^{\frac{1}{2}}$ to find x.

Ans. $x=4$, 9, or $\dfrac{-33\mp\sqrt{-67}}{2}$.

14. Given $(x-2)^2-6x^{\frac{1}{2}}(x-2)=24-14x+15x^{\frac{1}{2}}$ to find x.

Ans. $x=1$, 16, or $\dfrac{-1\pm 3\sqrt{-11}}{2}$.

15. Given $(4x+1)^2+4x^{\frac{1}{2}}(4x+1)=1912-(10x+3x^{\frac{1}{2}})$ to find x.

Ans. $x=9$, $12\frac{1}{4}$, or $\dfrac{-90\mp\sqrt{-181}}{8}$.

16. Given $x^2-\dfrac{27x}{4}+25=7\sqrt{x}(5-x)$ to find x.

Ans. $x=4$, $6\frac{1}{4}$, or $\dfrac{209\mp 13\sqrt{249}}{8}$.

17. Given $8x^2-13=\dfrac{3x}{2}+\sqrt{6x^3+52x^2}$ to find x.

Ans. $x=2$, $-1\frac{5}{8}$, or $\dfrac{3\pm\sqrt{3337}}{64}$.

18. Given $4x^2+21x+8x^{\frac{1}{2}}\sqrt{7x^2-5x}=207-\dfrac{4x^2}{3}$ to find x.

Ans. $x=3$, $12\frac{15}{16}$, or $\dfrac{-129\pm 3\sqrt{-2567}}{32}$.

19. Given $x^4-8x^3-12x^2+84x=63$ to find x.

Ans. $x=2\pm\sqrt{7}\pm\sqrt{11\pm\sqrt{7}}$.

20. Given $\dfrac{x^2+8}{6}-\dfrac{4x+6}{x^2}=\dfrac{8-\dfrac{2}{x^2}}{3}$ to find x.

Ans. $x=4$, -2, or $-1\pm\sqrt{-3}$.

21. Given $x^4-12x=5$ to find x. *Ans.* $x=1\pm\sqrt{2}$, or $-1\pm 2\sqrt{-1}$.

22. Given $x=\dfrac{4\sqrt{x}-48}{x-18}$ to find x. *Ans.* $x=4$. 16, or $8\mp 2\sqrt{7}$.

23. Given $x-\frac{8}{\sqrt{x}}-\frac{2}{x}=5\left(1+\frac{2}{x}\right)$ to find x.

Ans. $x=4,\ 9$, or $\frac{-3\mp\sqrt{-7}}{2}$.

24. Given $x^4-25x^2+60x-36=0$ to find x.

Ans. $x=1,\ 2,\ 3$, or -6.

25. Given $x^4-36x^2+72x-36=0$ to find x.

Ans. $x=3\pm\sqrt{3}$, or $-3\pm\sqrt{15}$.

26. Given $x^4-6x^2-8x-3=0$ to find x.

Ans. $x=3,\ -1,\ -1$, or -1.

27. Given $x^4-9x^2+4x+12=0$ to find x.

Ans. $x=2,\ 2,\ -1$, or -3.

28. Given $x^4-6x^3+24x-16=0$ to find x.

Ans. $x=\pm 2$, or $3\pm\sqrt{5}$.

29. Given $x^4-7x^3+59\frac{1}{2}x-72\frac{1}{4}=0$ to find x.

Ans. $x=3\frac{1}{2}-\pm\frac{1}{2}\sqrt{15}$.

30. Given $x^4-17x^2-20x-6=0$ to find x.

Ans. $x=2\pm\sqrt{7}$, or $-2\pm\sqrt{-2}$.

31. Given $x^4-55x^2-30x+504=0$ to find x.

Ans. $x=3,\ 7,\ -4$, or -6.

32. Given $x^4-3x^2-4x-3=0$ to find x.

Ans. $x=\frac{1\pm\sqrt{13}}{2}$, or $\frac{-1\pm\sqrt{-3}}{2}$.

33. Given $x^4-27x^2+14x+120=0$ to find x.

Ans. $x=3,\ 4,\ -2$, or -5.

34. Given $x^4+6x^3-24x-16=0$ to find x.

Ans. $x=2,\ -2$, or $-3\pm\sqrt{5}$.

35. Given $x^4-45x^2-40x+84=0$ to find x.

Ans. $x=1,\ 7,\ -2$, or -6.

36. Given $\sqrt[6]{\frac{1}{x^4}}+\sqrt[3]{\frac{1}{x}}=\frac{3-\sqrt[3]{x^2}}{x}$ to find x.

Ans. $x=\pm 1$, or $\pm\frac{27}{8}$.

37. Given $x^4\left(1+\frac{1}{3x}\right)^2-(3x^2+x)=70$ to find x.

Ans. $x=3,\ -3\frac{1}{3}$, or $\frac{-1\pm\sqrt{-251}}{6}$.

38. Given $\dfrac{35\frac{5}{7}}{\sqrt{x^4-9x^2}}+\dfrac{\sqrt{x^2-9}}{7x}=\dfrac{19}{2x}$ to find x.

Ans. $x=\pm 5$, or $\pm\frac{1}{2}\sqrt{15661}$.

39. Given $x^2-2x+4=2\sqrt{x^3-1}$ to find x.

Ans. $x=4\pm\sqrt{6}$, or $\pm\sqrt{-2}$.

40. Given $x-2\sqrt{x+2}=1+\sqrt[4]{x^3-3x+2}$ to find x.

Ans. $x=9\pm 4\sqrt{7}$, or $\dfrac{3\pm\sqrt{13}}{2}$.

(346.) An equation of the form $ax^4\pm bx^3\pm cx^2\pm bx+a=0$, is called a *recurring equation* of the fourth degree, or a *biquadratic recurring equation.*

PROBLEM.

(347.) Given $ax^4+bx^3+cx^2+bx+a=0$ (1) to find the values of x.

SOLUTION.

By multiplying by $4a$ we get

$$4a^2x^4+4abx^3+4acx^2+4abx+4a^2=0 \qquad (A)=(1)\times 4a.$$

$$(2ax^2+bx)^2+(4ac-b^2)x^2+4abx+4a^2=0 \qquad (B)$$

$$(2ax^2+bx)^2+4a(2ax^2+bx)+4a^2=(8a^2+b^2-4ac)x^2 \quad (C)$$

$$2ax^2+bx+2a=\pm x\sqrt{8a^2+b^2-4ac}$$

$$2ax^2+(b\mp\sqrt{8a^2+b^2-4ac})x=-2a$$

$$x=\frac{\pm\sqrt{8a^2+b-4ac}-b\pm\sqrt{-8a^2+2b^2-4ac\pm 2b\sqrt{8a^2+b^2-4ac}}}{4a}.$$

It may be perceived that the coefficient of x^2 in the term that is added to both members of (A), and which makes the first member a perfect square, is a function of known terms.

This coefficient is equal to the coefficient of x^2 in (B) subtracted from twice the coefficient x^4 in (A), or confining the explanation to the primitive equation, is equal to 8 times the square of the coefficient of x^4, plus the square of the coefficient of x^3, minus 4 times the product of the coefficient of x^4 by the coefficient of x^2.

If the primitive equation had been multiplied by a instead of $4a$, the coefficient of x^2 in the term added would have been just $\frac{1}{4}$ as much, or $\left(2a^2+\dfrac{b^2}{4}-ac\right)$.

NOTE.—This mode of treating recurring equations of the fourth degree is original. The method, it will be perceived, is a general one. The discovery was afterward independently made by M. C. Stevens, who, at our request, has furnished the following concisely written rule. In applying it, the original equation must not be multiplied by 4, as was done in our solution, in order to avoid fractions.

RULE.

Divide by the coefficient of x^4 and transpose the term containing x^2; then add to each member $2x^2+$ the square of half the term containing x, or $-2x^2+$ the same, according as the second and the fourth terms have like or unlike signs. Extract the square root and the equation reduces to the second degree.

(348.) The following is the plan usually given for the solution of a recurring equation of the fourth degree:

PROBLEM.

Given $ax^4+bx^3+cx^2+bx+a=0$ to find x.

SOLUTION.

$$ax^4+bx^3+cx^2+bx+a=0$$

$$ax^2+bx+c+\frac{b}{x}+\frac{a}{x^2}=0$$

$$ax^2+\frac{a}{x^2}+bx+\frac{b}{x}+c=0$$

$$a\left(x^2+\frac{1}{x^2}\right)+b\left(x+\frac{1}{x}\right)+c=0$$

Now put $\qquad x+\frac{1}{x}=y$

Then $\qquad x^2+\frac{1}{x}=y^2-2$

Substituting $a(y^2-2)+by+c=0$

$$ay^2+by=2a-c$$

$$y=\frac{-b\pm\sqrt{8a^2+b^2-4ac}}{2a}$$

$$\therefore\quad x+\frac{1}{x}=\frac{-b\pm\sqrt{8a^2+b^2-4ac}}{2a}$$

$$2ax^2+(b\mp\sqrt{8a^2+b^2-4ac})x=-2a$$

$$x=\frac{\pm\sqrt{8a^2+b^2-4ac}-b\pm\sqrt{-8a^2+2b^2-4ac\pm2b\sqrt{8a^2+b^2-4ac}}}{4a}.$$

PROBLEM.

Given $10x^4-9x^3-20x^2-9x+10=0$ to find the values of x.

SOLUTION 1.

$$10x^4-9x^3-20x^2-9x+10=0,$$
$$400x^4-360x^3-800x^2-360x+400=0,$$
$$(20x^2-9x)^2-881x^2-360x+400=0,$$
$$(20x^2-9x)^2+40(20x^2-9x)+400=1681x^2.$$
$$20x^2-9x+20=\pm 41x,$$
$$\therefore\quad 20x^2-50x=-20;\qquad \text{or } 20x^2+32x=-20,$$
$$2x^2-5x=-2\qquad\qquad 5x^2+8x=-5,$$
$$x=\frac{5\pm 3}{4}=2, \text{ or } \tfrac{1}{2}\qquad x=\frac{-4\pm 3\sqrt{-1}}{5}.$$

SOLUTION 2.

$$10x^4-9x^3-20x^2-9x+10=0,$$
$$10x^2-9x^2-20-\frac{9}{x}+\frac{10}{x^2}=0,$$
$$10\left(x^2+\frac{1}{x^2}\right)-9\left(x+\frac{1}{x}\right)=20,$$
$$\text{put}\quad x+\frac{1}{x}=y,$$
$$\text{then}\quad x^2+\frac{1}{x^2}=y^2-2,$$
$$\text{substituting } 10y^2-20-9y=20,$$
$$10y^2-9y=40,$$
$$y=\frac{9\pm 41}{20}=\frac{5}{2}, \text{ or } -\frac{8}{5},$$
$$\therefore\quad x+\frac{1}{x}=\frac{5}{2}, \text{ or } -\frac{8}{5},$$
$$\text{and}\quad 2x^2-5x=-2;\qquad \text{or } 5x^2+8x=-5,$$
$$x=\frac{5\pm 3}{4}=2, \text{ or } \tfrac{1}{2}\qquad x=\frac{-4\pm 3\sqrt{-1}}{5}.$$

EXAMPLES.

1. Given $x^4+24x^3-114x^2-24x+1=0$ to find x.

Ans. $x=2\pm\sqrt{5}$, or $-14\pm\sqrt{197}$.

2. Given $x^4+5x^3+2x^2+5x+1=0$ to find x.

Ans. $x=\pm\sqrt{-1}$, or $\dfrac{-5\pm\sqrt{21}}{2}$.

3. Given $x^4+x^3+x^2+x+1=0$ to find x.

$-1+\sqrt{5}+\sqrt{-10+2\sqrt{5}}$

$$Ans.\ x=\frac{-1\pm\sqrt{5}\pm\sqrt{-10\mp2\sqrt{5}}}{4}.$$

$- \quad + \quad - \quad +$

4. Given $2x^4-4x^3-6x^2-4x+2=0$ to find x.

$- \quad - \quad + \quad -$

$- \quad - \quad - \quad -$

$$Ans.\ x=\frac{1\pm\sqrt{6}\pm\sqrt{3\pm2\sqrt{6}}}{2}.$$

5. Given $3x^4+2x^3+4x^2+2x+3=0$ to find x.

$$Ans.\ x=\frac{-1\pm\sqrt{7}\pm\sqrt{-28\mp2\sqrt{7}}}{6}.$$

6. Given $4x^4+3x^3-8x^2-3x+4=0$ to find x.

$$Ans.\ x=\pm1, \text{ or } \frac{-3\pm\sqrt{73}}{8}.$$

7. Given $\frac{1+x^4}{(1+x)^4}=\frac{1}{2}$ to find x.

$$Ans.\ x=1\pm\sqrt{3}\pm\sqrt{3}\sqrt{\sqrt{3}\pm2}.$$

(349.) There are other biquadratics that are not recurring which are susceptible of a similar solution, but the coefficient of x^2 must be decided by trial.

EXAMPLES.

1. Given $x^4-2x^3-7x^2-8x+16=0$ to find x.

$$Ans.\ x=1,\ 4, \text{ or } \frac{-3\pm\sqrt{-7}}{2}.$$

2. Given $x^4+x^3-x^2+2x+4=0$ to find x.

$$Ans.\ x=\frac{-1\pm\sqrt{21}\pm\sqrt{-10\mp2\sqrt{21}}}{4}.$$

3. Given $4x^4+8x^3-89x^2+28x+49=0$ to find x.

$$Ans.\ x=1,\ 3\tfrac{1}{2}, \text{ or } \frac{-13\pm\sqrt{113}}{4}.$$

4. Given $2x^4+24x^3-315x^2+216x+162=0$ to find x.

$$Ans.\ x=-3\pm\tfrac{3}{4}\sqrt{94}\pm3\sqrt{\tfrac{47}{8}\mp\tfrac{1}{2}\sqrt{94}}.$$

5. Given $x^4-12x^3+47x^2-72x+36=0$ to find x.

$$Ans.\ x=1,\ 2,\ 3, \text{ or } 6.$$

6. Given $x^4-9x^3+15x^2-27x+9=0$ to find x.

$$Ans.\ x=\frac{9\pm3\sqrt{5}\pm\sqrt{78\pm54\sqrt{5}}}{4}.$$

7. Given $x^4+36x^3-400x^2-3168x+7744=0$ to find x.

Ans. $x=-9\pm\sqrt{137}\pm\sqrt{306\mp18\sqrt{137}}$.

8. Given $\sqrt{x}-\frac{8}{x}=\frac{7}{\sqrt{x}-2}$ to find x. *Ans.* $x=1$, 16, or $\frac{7\pm3\sqrt{-7}}{2}$.

(350.) There is a class of problems which may be solved after the manner given in the solution to the following

PROBLEM.

Given $x^4+2x^3-7x^2-8x=-12$ to find the value of x.

SOLUTION.

$$x^4+2x^3-7x^2-8x=-12,$$
$$(x^2+x)^2-8x^2-8x=-12,$$
$$(x^2+x)^2-8(x^2+x)=-12,$$
$$(x^2+x)^2-8(x^2+x)+16=4,$$
$$x^2+x-4=\pm2,$$
$$x^2+x=6, \text{ or } 2,$$
$$x=2, -3, 1, \text{ or } -2.$$

EXAMPLES.

1. Given $x^4+2x^3-3x^2-4x+4=0$ to find x.

Ans. $x=1, 1, -2$, or -2.

2. Given $x^4-12x^3+50x^2-84x+49=0$ to find x.

Ans. $x=3\pm\sqrt{2}$, or $3\pm\sqrt{2}$.

3. Given $x^4-10x^3+35x^2-50x+24=0$ to find x.

Ans. $x=1, 2, 3$, or 4.

4. Given $x^4+2x^3-13x^2-14x+24=0$ to find x.

Ans. $x=1, 3, -2$, or -4.

5. Given $x^4+12x^3+54x^2+108x+81=0$ to find x.

Ans. $x=-3, -3, -3$, or -3.

6. Given $x^4+2qx^3+3q^2x^2+2q^3x=r^4$ to find x.

Ans. $x=\frac{-q\pm\sqrt{-3q^2\pm4\sqrt{r^4+q^4}}}{2}$.

7. Given $x^4-14x^3+61x^2-84x+36=0$ to find x.

Ans. $x=1, 1, 6$, or 6.

8. Given $4x^4+\frac{x}{2}=4x^3+33$ to find x.

Ans. $x=2$, $-1\frac{1}{2}$, or $\frac{1\pm\sqrt{-43}}{4}$.

9. Given $x^4-2x^3-2x^2+3x=108$ to find x.

Ans. $x=4,\ -3$, or $\dfrac{1\pm\sqrt{-35}}{2}$.

10. Given $x^4-2x^3+x=30$ to find x.

Ans. $x=3,\ -2$, or $\dfrac{1\pm\sqrt{-19}}{2}$.

11. Given $x^4-6x^3+5x^2+12x=60$ to find x.

Ans. $x=5,\ -2$, or $\dfrac{3\pm\sqrt{-15}}{2}$.

12. Given $x^4-8x^3+10x^2+24x+5=0$ to find x.

Ans. $x=5,\ -1$, or $2\pm\sqrt{5}$.

13. Given $x^4-2x^3+x=132$ to find x.

Ans. $x=4,\ -3$, or $\dfrac{1\pm\sqrt{-43}}{2}$.

14. Given $x^4-2ax^3+(a^2-2)x^2+2ax=a^2$ to find x.

Ans. $x=\dfrac{a}{2}\pm\sqrt{\dfrac{a^2}{4}+1\pm\sqrt{1+a^2}}$.

15. Given $x^4-8ax^3+8a^2x^2+32a^3x=9a^4$ to find x.

Ans. $x=2a\pm a\sqrt{3}$, or $2a\pm a\sqrt{13}$.

16. Given $\dfrac{18}{x^2}+\dfrac{81-x^2}{9x}=\dfrac{x^2-65}{72}$ to find x.

Ans. $x=9,\ -9,\ -4$, or -4.

17. Given $x^4-2x^3-25x^2+26x+120=0$ to find x.

Ans. $x=3,\ 5,\ -2$, or -4.

18. Given $x^4-12x^3+44x^2-48x=9009$ to find x.

Ans. $x=13,\ -7$, or $3\pm3\sqrt{-10}$.

19. Given $x^2-2x^{\frac{3}{2}}+2x-\sqrt{x}=6$ to find x.

Ans. $x=1,\ 4$, or $\dfrac{-5\pm\sqrt{-11}}{2}$.

20. Given $x^4-6x^3+13x^2-12x=5$ to find x.

Ans. $x=\dfrac{3\pm\sqrt{13}}{2}$, or $\dfrac{3\pm\sqrt{-11}}{2}$.

21. Given $x^4-8ax^3+8a^2x^2+32a^3x=d$ to find x.

Ans. $x=2a\pm\sqrt{8a^2\pm\sqrt{16a^4+d}}$.

22. Given $x^4-4x^3+8x^2-8x=21$ to find x.

Ans. $x=3,\ -1$, or $1\pm\sqrt{-6}$.

23. Given $4x^4+24x^3+52x^2+48x=480$ to find x.

$$Ans.\ x=\frac{-3\pm\sqrt{1\pm8\sqrt{31}}}{2}.$$

24. Given $x^4+12x^3+40x^2+24x=837$ to find x.

$$Ans.\ x=3,\ -9,\ \text{or}\ -3\pm\sqrt{-22}.$$

25. Given $x^4+6x^3+80x^2+213x=2128$ to find x.

$$Ans.\ x=\frac{-3\pm\sqrt{-133\pm2\sqrt{13553}}}{2}.$$

(351.) The following examples are best solved by factoring, since the factors are readily obtained. The solution of the following problem will serve as an illustration.

PROBLEM.

Given $x^4+3x^3-3x=9$ to find the values of x.

SOLUTION.

$$x^4+3x^3-3x=9,$$
$$x^4+3x^3=3x+9,$$
$$(x+3)x^3=3(x+3),$$
$$x^3=3,$$
$$x=\sqrt[3]{3},$$

Dividing x^3-3 by $x-\sqrt[3]{3}$, we get $x^2+\sqrt[3]{3}x+\sqrt[3]{9}=0$,

$$x^2+\sqrt[3]{3}x=-\sqrt[3]{9},$$
$$x=\frac{-\sqrt[3]{3}\pm\sqrt{-3\sqrt[3]{9}}}{2},$$

and $x+3=0$,

$$x=-3.$$

EXAMPLES.

1. Given $x^4+\dfrac{13x^3}{3}-39x=81$ to find x.

$$Ans.\ x=\pm3,\ \text{or}\ \frac{-13\pm\sqrt{-155}}{6}.$$

2. Given $25x^4-624x^2=25$ to find x.

$$Ans.\ x=\pm5,\ \text{or}\ \pm\tfrac{1}{5}\sqrt{-1}.$$

3. Given $x^4-x^3-2x=4$ to find x.

$$Ans.\ x=2,\ -1,\ \text{or}\ \pm\sqrt{-2}.$$

BIQUADRATIC EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

(352.) In eliminating one of the unknown quantities, the student must be guided by the methods used in equations of the lower degrees, adopting that mode which seems to be best suited to the particular problem under consideration. In the following examples there are some which belong to what are called

HOMOGENEOUS EQUATIONS.

(353.) *Homogeneous Equations* of the fourth degree are those in which each of the literal terms contains two literal factors; as

$$x^2+xy=a,$$
$$y^2+x^2=b,$$

for these equations are the same as when they are written $xx+xy=a$ and $yy+xx=b$, in which it is seen that each literal term contains two literal factors. Homogeneous equations of the fourth degree are susceptible of a complete solution, according to the plan exhibited in the solution of the following

PROBLEM.

Given $\left\{\begin{array}{l} x^2+xy=56, \\ xy+2y^2=60, \end{array}\right\}$ to find x and y.

SOLUTION.

$$x^2+xy=56,$$
$$xy+2y^2=60,$$

Put $y=nx$ $\quad x^2+nx^2=56,$

$$x^2=\frac{56}{1+n},$$

and $\quad nx^2+2n^2x^2=60,$

$$x^2=\frac{60}{n+2n^2},$$
$$\frac{56}{1+n}=\frac{60}{n+2n^2},$$
$$\frac{14}{1+n}=\frac{15}{n+2n^2},$$
$$28n^2+14n=15+15n,$$
$$28n^2-n=15,$$
$$n=\frac{1\pm41}{56}=\tfrac{3}{4}, \text{ or } -\tfrac{5}{7}.$$

Whence, $\quad x=\pm4\sqrt{2}$, or ±14,

and $\quad y=\pm3\sqrt{2}$, or ±10.

REMARK.—Some of the following examples are composed of homogeneous equations, and may, therefore, be solved after the manner of the one just given. The others are miscellaneous in their character, and various are the artifices to be employed in reducing them. The student, as a general rule, should so manage the equation as to obtain a perfect square in the left hand member, and this may be obtained frequently before elimination.

EXAMPLES.

1. Given $\left\{\begin{array}{l}(x+y)^2:(x-y)^2::64:1,\\ xy=63,\end{array}\right\}$ to find x and y.

Ans. $x=\pm 9$, or ± 7, $y=\pm 7$, or ± 9.

2. Given $\left\{\begin{array}{l}xy^2+y\ \ =21,\\ x^2y^4+y^2=333,\end{array}\right\}$ to find x and y.

Ans. $\left\{\begin{array}{l}x=\pm 2, \text{ or } \pm\frac{1}{108},\\ y=3, \text{ or } 18.\end{array}\right.$

3. Given $\left\{\begin{array}{l}x+\sqrt{xy}+y=19,\\ x^2+xy+y^2=133,\end{array}\right\}$ to find x and y.

Ans. $\left\{\begin{array}{l}x=\pm 9, \text{ or } \pm 4,\\ y=\pm 4, \text{ or } \pm 9.\end{array}\right.$

4. Given $\left\{\begin{array}{r}x+y+\sqrt{x+y}=\ \ 6,\\ x^2+y^2=10,\end{array}\right\}$ to find x and y.

Ans. $\left\{\begin{array}{l}x=1,\ 3, \text{ or } \dfrac{9\pm\sqrt{-61}}{2},\\ y=1,\ 3, \text{ or } \dfrac{9\pm\sqrt{-61}}{2}.\end{array}\right.$

5. Given $\left\{\begin{array}{c}\dfrac{y}{(x+y)^{\frac{3}{2}}}+\dfrac{\sqrt{x+y}}{y}=\dfrac{17}{4\sqrt{x+y}},\\ x=y^2+2,\end{array}\right\}$ to find x and y.

Ans. $\left\{\begin{array}{l}x=3,\ 6, \text{ or } \dfrac{9\pm 3\sqrt{-119}}{32},\\ y=1,\ 2, \text{ or } \dfrac{-3\pm\sqrt{-119}}{8}.\end{array}\right.$

6. Given $\left\{\begin{array}{l}3x^2+xy\ \ =68,\\ 4y^2+3xy=160,\end{array}\right\}$ to find x and y.

Ans. $\left\{\begin{array}{l}x=\pm 4, \text{ \&c.,}\\ y=\pm 5, \text{ \&c.}\end{array}\right.$

7. Given $\left\{\begin{array}{r}x^2+x+y=18-y^2\\ xy=6,\end{array}\right\}$ to find x and y.

Ans. $\left\{\begin{array}{l}x=3,\ 2, \text{ or } -3\pm\sqrt{3},\\ y=2,\ 3, \text{ or } -3\mp\sqrt{3}.\end{array}\right.$

8. Given $\left\{\begin{array}{r} x^2+2xy+y^2+2y=120-2x, \\ xy-y^2=8, \end{array}\right\}$ to find x and y.

Ans. $\left\{\begin{array}{l} x=6,\ 9, \text{ or } -9\mp\sqrt{5}, \\ y=4,\ 1, \text{ or } -3\pm\sqrt{5}. \end{array}\right.$

9. Given $\left\{\begin{array}{r} x^2+y^2-x-y=78, \\ xy+x+y=39, \end{array}\right\}$ to find x and y.

Ans. $\left\{\begin{array}{l} x=9,\ 3, \text{ or } \dfrac{-13\pm\sqrt{-39}}{2}, \\ y=3,\ 9, \text{ or } \dfrac{-13\mp\sqrt{-39}}{2}. \end{array}\right.$

10. Given $\left\{\begin{array}{l} 4xy=96-x^2y^2, \\ x+y=6, \end{array}\right\}$ to find x and y.

Ans. $\left\{\begin{array}{l} x=4,\ 2, \text{ or } 3\pm\sqrt{21}, \\ y=2,\ 4, \text{ or } 3\mp\sqrt{21}. \end{array}\right.$

11. Given $\left\{\begin{array}{r} x^2y^4-7xy^2-945=765, \\ xy-y=12, \end{array}\right\}$ to find x and y.

Ans. $\left\{\begin{array}{l} x=5,\ \frac{1}{5}, \text{ or } \dfrac{-19}{17\mp6\sqrt{-2}}, \\ y=3,\ -15, \text{ or } -6\pm\sqrt{-2}. \end{array}\right.$

12. Given $\left\{\begin{array}{r} x^2+y^2=5, \\ xy=2, \end{array}\right\}$ to find x and y.

Ans. $\left\{\begin{array}{l} x=2,\ 1,\ -2, \text{ or } -1, \\ y=1,\ 2,\ -1, \text{ or } -2. \end{array}\right.$

13. Given $\left\{\begin{array}{r} x^2+y^2=s^2, \\ xy=a^2, \end{array}\right\}$ to find x and y.

Ans. $\left\{\begin{array}{l} x=\frac{1}{2}(\pm\sqrt{s^2+2a^2}\pm\sqrt{s^2-2a^2}), \\ y=\frac{1}{2}(\pm\sqrt{s^2+2a^2}\mp\sqrt{s^2-2a^2}). \end{array}\right.$

14. Given $\left\{\begin{array}{l} \sqrt{\dfrac{3x}{x+y}}+\sqrt{\dfrac{x+y}{3x}}=2, \\ xy-(x+y)=54, \end{array}\right\}$ to find x and y.

Ans. $\left\{\begin{array}{l} x=\ 6,\ -4\frac{1}{2},\ 6, \text{ or } -4\frac{1}{2}, \\ y=12,\ -9,\ 12, \text{ or } -9. \end{array}\right.$

15. Given $\left\{\begin{array}{l} 6x^2+2y^2=5xy+12, \\ 2xy+3x^2=3y^2-3, \end{array}\right\}$ to find x and y.

Ans. $\left\{\begin{array}{l} x=\pm2, \text{ \&c.}, \\ y=\pm3, \text{ \&c.} \end{array}\right.$

16. Given $\left\{\begin{array}{l} x^2y+xy^2=30, \\ \dfrac{1}{x}+\dfrac{1}{y}\ =\frac{5}{6}, \end{array}\right\}$ to find x and y.

Ans. $\left\{\begin{array}{l} x=3,\ 2,\ 1, \text{ or } -6, \\ y=2,\ 3,\ -6, \text{ or } 1. \end{array}\right.$

17. Given $\left\{\begin{array}{c} \dfrac{x^2}{y^2}+\dfrac{y}{x}+\dfrac{x}{y}=6\tfrac{3}{4}-\dfrac{y^2}{x^2}, \\ x-y=2, \end{array}\right\}$ to find x and y.

Ans. $\left\{\begin{array}{l} x=4,\ -2, \text{ or } 1\pm\sqrt{\tfrac{3}{11}}, \\ y=2,\ -4, \text{ or } -1\pm\sqrt{\tfrac{3}{11}}. \end{array}\right.$

18. Given $\left\{\begin{array}{c} x^2+y^2+x-y=132, \\ (x^2+y^2)(x-y)=1220, \end{array}\right\}$ to find x and y.

Ans. $\left\{\begin{array}{l} x=11,\ -1, \text{ or } 61\pm\sqrt{-3716}, \\ y=1,\ -11, \text{ or } -61\pm\sqrt{-3716}. \end{array}\right.$

19. Given $\left\{\begin{array}{r} \dfrac{x}{y}-8\sqrt{x^2-9xy^2}=9y-16xy, \\ 5x=4+25y^2, \end{array}\right\}$ to find x and y.

Ans. $\left\{\begin{array}{l} x=1,\ 1,\ -\tfrac{9}{20}, \text{ or } -\tfrac{9}{20}, \\ y=\tfrac{1}{5},\ -\tfrac{1}{5},\ \tfrac{1}{2}\sqrt{-1}, \text{ or } -\tfrac{1}{2}\sqrt{-1}. \end{array}\right.$

20. Given $\left\{\begin{array}{r} x+y+\sqrt{x+y}=12, \\ x^3+y^3=189, \end{array}\right\}$ to find x and y.

Ans. $\left\{\begin{array}{l} x=5,\ 4, \text{ or } 8\pm\tfrac{1}{4}\sqrt{-\tfrac{835}{3}}, \\ y=4,\ 5, \text{ or } 8\mp\tfrac{1}{4}\sqrt{-\tfrac{835}{3}}. \end{array}\right.$

21. Given $\left\{\begin{array}{l} y+\sqrt{\dfrac{y}{x}}=\dfrac{42}{x}, \\ \dfrac{x^2}{3}+\dfrac{x}{2\sqrt{y}}=\dfrac{54}{y}, \end{array}\right\}$ to find x and y.

Ans. $\left\{\begin{array}{l} x=4,\ \tfrac{144}{49},\ \tfrac{81}{16}, \text{ or } \tfrac{729}{196}, \\ y=9,\ \tfrac{49}{4},\ \tfrac{784}{81}, \text{ or } \tfrac{49136}{729}. \end{array}\right.$

22. Given $\left\{\begin{array}{l} (x^2+1)y=xy+126, \\ (x^2+1)y=x^2y^2-744, \end{array}\right\}$ to find x and y.

Ans. $\left\{\begin{array}{l} x=5,\ \tfrac{1}{5}, \text{ or } \dfrac{-97\pm\sqrt{6045}}{58}, \\ y=6,\ 150, \text{ or } \dfrac{1682}{97\mp\sqrt{6045}}. \end{array}\right.$

23. Given $\left\{\begin{array}{r} xy=125x+300y, \\ y^2-x^2=90000, \end{array}\right\}$ to find x and y.

Ans. $\left\{\begin{array}{l} x=400,\ 225, \text{ or } 12\tfrac{1}{2}(-1-5\sqrt{-23}, \\ y=500,\ -375, \text{ or } \dfrac{125\pm25\sqrt{-23}}{2}. \end{array}\right.$

24. Given $\begin{cases} x+y=3, \\ x^4+y^4=17, \end{cases}$ to find x and y.

$$Ans. \begin{cases} x=2,\ 1,\ \dfrac{3\pm\sqrt{-55}}{2}, \\ y=1,\ 2,\ \dfrac{3\mp\sqrt{-55}}{2}. \end{cases}$$

25. Given $\begin{cases} 8x+23y=2x^3+2y^3, \\ 34y+6x^2=5y^2+13xy+24, \end{cases}$ to find x and y.

$$Ans. \begin{cases} x=3,\ -\dfrac{171}{133},\ \text{or}\ \dfrac{55\mp\sqrt{1114}}{26}, \\ y=2,\ \dfrac{34}{133},\ \text{or}\ \dfrac{-9\pm3\sqrt{1114}}{26}. \end{cases}$$

(354.) Biquadratic equations which do not admit of solution by any of the previous methods may frequently be solved as cubics by the addition of a binomial squared to both members.

PROBLEM.

Given $x^4+4x^3-x^2-16x=12$ to find the values of x.

SOLUTION.

$$x^4+4x^3-x^2-16x=12$$
$$(x^2+2x)^2-5x^2-16x=12$$
$$(x^2+2x)^2-4(x^2+2x)+4=x^2+8x+16.$$
$$x^2+2x-2=x+4,\ \text{or}\ -x-4,$$
$$\therefore\quad x^2+x=6\qquad;\ \text{or}\ x^2+3x=-2$$
$$x=\frac{-1\pm5}{2}=2,-3\qquad x=\frac{-3\pm1}{2}=-1,-2$$

EXAMPLES.

1. Given $x^4-6x^3+12x^2-10x+3=0$ to find x.

Ans. $x=1$, 1, 1, or 3.

2. Given $x^4-4x^3-19x^2+46x+120=0$ to find x.

Ans. $x=4$, 5, -2, or -3.

3. Given $x^4+3x^3+x^2-3x=2$ to find x.

Ans. $x=1$, -1, -1, or -2.

4. Given $x^4-6x^3+5x^2+2x^2=10$ to find x.

Ans. $x=5$, -1, or $1\pm\sqrt{-1}$.

5. Given $x^4-4x^3-8x+32=0$ to find x.

Ans. $x=2$, 4, or $-1\pm\sqrt{-3}$.

6. Given $x^4-9x^3+30x^2-46x+24=0$ to find x.

Ans. $x=1$, 4, or $2\pm\sqrt{-2}$.

7. Given $x^4-x^3+2x^2+x=3$ to find x.

Ans. $x=\pm\bar{1}$, or $\frac{1\pm\sqrt{-11}}{2}$.

8. Given $6x^4-43x^3+107x^2-108x+36=0$ to find x.

Ans. $x=\frac{2}{3}$, $1\frac{1}{2}$, 2, or 3.

9. Given $x^4+x^3-16x^2-4x+48=0$ to find x.

Ans. $x=2$, 3, -2, or -4.

10. Given $x^4-x^3-11x^2+9x+18=0$ to find x.

Ans. $x=2$, 3, -1, or -3.

11. Given $x^4-8x^3+14x^2+4x=8$ to find x.

Ans. $x=3\pm\sqrt{5}$, or $1\pm\sqrt{3}$.

12. Given $x^4-12x^3+48x^2-68x+15=0$ to find x.

Ans. $x=3$, 5, or $2\pm\sqrt{3}$.

13. Given $2x^4-2x^3-2x^2+\frac{3x}{2}+\frac{3}{8}=0$ to find x.

Ans. $x-\pm\frac{1}{2}\sqrt{3}$, or $\frac{1\pm\sqrt{2}}{2}$.

14. Given $x^4+x^3-29x^2-9x+180=0$ to find x.

Ans. $x=3$, 4, -3, or -5.

15. Given $x^4-4x^3-29x^2+156x=180$ to find x.

Ans. $x=2$, 3, 5, or -6.

16. Given $x^4+29x^3+287x^2+1147x+1560=0$ to find x.

Ans. $x=-3$, -5, -8, or -13.

17. Given $x^4-9x^3+\frac{45x^2}{4}+\frac{27x}{2}=\frac{81}{4}$ to find x.

Ans. $x=1\frac{1}{2}$, $1\frac{1}{2}$, or $3\pm3\sqrt{2}$.

18. Given $x^4-8x^3+23x^2-64x+120=0$ to find x.

Ans. $x=3$, 5, or $\pm2\sqrt{-2}$.

19. Given $x^4-16x^3+79x^2-140x+58=0$ to find x.

Ans. $x=2\pm\sqrt{2}$, or $6\pm\sqrt{7}$.

20. Given $x^4-5x^3-5x^2+45x=36$ to find x.
Ans. $x=1, 3, 4$, or -3.

21. Given $x^4-3x^3-15x^2+49x=12$ to find x.
Ans. $x=3, -4$, or $2\pm\sqrt{3}$.

22. Given $x^4+x^3-x^2-5x+4=0$ to find x.
Ans. $x=1, 1$, or $\dfrac{-3\pm\sqrt{-7}}{2}$.

23. Given $x^4+x^3-x^2+10x+4=0$ to find x.
Ans. $x=\dfrac{-3\pm\sqrt{5}}{2}$, or $1\pm\sqrt{-3}$.

24. Given $x^4-7x^3+9x^2+27x=54$ to find x.
Ans. $x=3, 3, 3$, or -2.

25. Given $x^4+3x^3-7x^2-27x=18$ to find x.
Ans. $x=3, -1, -2$, or -3.

26. Given $6x^4-25x^3+26x^2+4x=8$ to find x.
Ans. $x=\frac{2}{3}, 2, 2$, or $-\frac{1}{2}$.

27. Given $8x^4-38x^3+49x^2-22x+3=0$ to find x.
Ans. $x=\frac{1}{4}, \frac{1}{2}, 1$, or 3.

28. Given $x^4-9x^3+17x^2+27x=60$ to find x.
Ans. $x=4, 5$, or $\pm\sqrt{3}$.

29. Given $x^4+x^3-24x^2+43x=21$ to find x.
Ans. $x=1, 3$, or $\dfrac{-5\pm\sqrt{53}}{2}$.

30. Given $x^4+x^3+x^2-120x=100$ to find x.
Ans. $x=2\pm2\sqrt{2}$, or $\dfrac{-5\pm5\sqrt{-5}}{2}$.

31. Given $x^4+x^3+x^2+141x=100$ to find x.
Ans. $x=\dfrac{-5\pm\sqrt{41}}{2}$, or $2\pm\sqrt{-21}$.

32. Given $x^4+x^3-19x^2-49x=30$ to find x.
Ans. $x=5, -1, -2$, or -3.

33. Given $x^4-x^3-19x^2+49x=30$ to find x.
Ans. $x=1, 2, 3$, or -3.

34. Given $x^4-\frac{19}{4}x^3+\frac{49}{8}x^2-\frac{11}{4}x+\frac{3}{8}=0$ to find x.
Ans. $x=\frac{1}{4}, \frac{1}{2}, 1$, or 3.

35. Given $x^4-38x^3+210x^2+538x+289=0$ to find x.
Ans. $x=-1, -1$, or $20\pm\sqrt{111}$.

36. Given $4x^4-14x^3-5x^2+31x+6=0$ to find x.

Ans. $x=2,\ 3,\ \text{or}\ \dfrac{-3\pm\sqrt{5}}{4}$.

37. Given $x^4-6x^3-58x^2-114x=11$ to find x.

Ans. $x=\frac{3}{2}\pm\frac{5}{2}\sqrt{3}\pm\sqrt{17\pm\frac{21}{2}\sqrt{3}}$.

(355.) This method of solution is applicable to all affected biquadratic equations, as is shown by the solution of the following literal equation. But it is not always practicable, as the quantity to be added is frequently of such a character that it can not be easily found.

PROBLEM.

Given $x^4-(a+b+c+d)x^3+(ab+ac+ad+bc+bd+cd)x^2-(abc+abd+acd+bcd)x+abcd=0$ to find x.

SOLUTION.

$$4x^4-4(a+b+c+d)x^3+4(ab+ac+ad+bc+bd+cd)x^2-4(abc+abd+acd+bcd)x+4abcd=0,$$

$$[2x^2-(a+b+c+d)x]^2-(a^2-2ab-2ac-2ad+b^2-2bc-2bd+c^2-2cd+d^2)x^2-4(abc+abd+acd+bcd)x+4abcd=0,$$

$$[2x^2-(a+b+c+d)x]^2+2(ab+cd)\left(2x^2-(a+b+c+d)x\right)+a^2b^2+2abcd+c^2d^2=(a+b-c-d)^2x^2-2(a+b-c-d)(ab-cd)x+a^2b^2-2abcd+c^2d^2,\quad \therefore$$

$$2x^2-(a+b+c+d)x+ab+cd=(a+b-c-d)x-(ab-cd),\ \text{or}\ ab-cd-(a+b-c-d)x,$$

$$2x^2-2(a+b)x=-2ab;\quad \text{or}\ 2x^2-2(c+d)x=-2cd,$$

$$x=\frac{a+b\pm(a-b)}{2}=a,\ \text{or}\ b \qquad x=\frac{c+d\pm(c-d)}{2}=c,\ \text{or}\ d.$$

A close inspection of this solution will enable the student to see what relation the coefficient of x^2, in the added square, bears to the values of x, as finally ascertained.

MISCELLANEOUS QUESTIONS.*

1. A vintner draws a certain quantity of wine out of a full vessel that holds 256 gallons; and then filling the vessel with water, draws

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* These questions should be solved without resorting to the method just given.

off the same quantity of liquor as before, and so on, for four draughts, when there were only 81 gallons of pure wine left. How much wine did he draw each time? *Ans.* 64, 48, 36, and 27 gallons.

2. An upholsterer has 2 square carpets divided into square yards by the lines of the pattern. Now, he observes that if he subtracts from the number of squares in the smaller carpet, the number of yards in the side of the other, the square of the remainder will exceed the difference of the number of squares in the smaller carpet, and the number of yards in its side, by 88. Also, the difference of the lengths of the sides of the carpets is 6 feet. What is the size of each carpet?

Ans. 16 and 36 square yards.

3. A man, playing at hazard, won at the first throw as much money as he had in his pocket; at the second throw he won 5 shillings more than the square root of what he then had; at the third throw he won the square of all he then had, and then he had £112 16*s*. How much had he at first?

Ans. 18, or $24\frac{1}{2}$ shillings.

CHAPTER XIV.

HIGHER EQUATIONS.

(356.) EQUATIONS of the *fifth degree*, formerly called ***sursolid equations*** and equations of higher degrees, have not as yet been found to be susceptible of any general solution. Particular examples, however, frequently occur that may be reduced by known methods. It is the object of this chapter to present some of them.

PROBLEM

1. Given $x^6=a^6$ to find the values of x.

SOLUTION I.

$$x^6=a^6,$$
$$(x^3+a^3)(x^3-a^3)=0.$$
$$\therefore \quad x^3+a^3=0.$$
$$(x+a)(x^2-ax+a^2)=0,$$
$$x^2-ax+a^2=0,$$
$$x=\frac{a\pm a\sqrt{-3}}{2}.$$
$$x+a=0,$$
$$x=-a,$$
$$x^3-a^3=0,$$
$$(x-a)(x^2+ax+a^2)=0,$$
$$x^2+ax+a^2=0,$$
$$x=\frac{-a\pm a\sqrt{-3}}{2}.$$
$$x-a=0,$$
$$x=a,$$
$$\therefore \quad x=a,\ -a,\ \frac{a\pm a\sqrt{-3}}{2},\text{ or }\frac{-a\pm a\sqrt{-3}}{2}.$$

SOLUTION II.

$$x^6=a^6,$$

$$x^6-a^6=0,$$

$$(x-a)(x^5+ax^4+a^2x^3+a^3x^4+a^4x+a^5)=0,$$

$$x-a=0,$$

$$x=a,$$

$$x^5+ax^4+a^2x^3+a^3x^2+a^4x+a^5=0,$$

$$(x+a)x^4+a^2x^2(x+a)+a^4(x+a)=0,$$

$$(x+a)(x^4+a^2x^2+a^4)=0,$$

$$x+a=0,$$

$$x=-a,$$

$$x^4+a^2x^2+a^4=0,$$

$$x^2=\frac{-a^2\pm a^2\sqrt{-3}}{2}.$$

Applying the rules for finding the square root of surds, we get

$$x=\frac{a\pm a\sqrt{-3}}{2}, \text{ or } \frac{-a\pm a\sqrt{-3}}{2}.$$

It is easier, however, to get these values of x by considering

$$x^4+a^2x^2+a^4=(x^2+a^2)^2-a^2x^2$$

$$\therefore \quad (x^2+a^2)^2-a^2x^2=0,$$

$$(x^2+a^2-ax)(x^2+a^2+ax)=0,$$

$$x^2-ax+a^2=0,$$

$$\text{and} \quad x^2+ax+a^2=0,$$

$$x=\frac{a\pm a\sqrt{-3}}{2}.$$

$$\text{and} \qquad x=\frac{-a\pm a\sqrt{-3}}{2}.$$

PROBLEM

2. Given $x^5=a^5$ to find x.

SOLUTION.

$$x^5=a^5$$

$$x=a$$

$$\therefore \quad (x-a)(x^4+ax^3+a^2x^2+a^3x+a^4)=0.$$

Placing $x^4+ax^3+a^2x^2+a^3x+a^4=0$, we have a biquadratic equation in which the coefficients are literal. To solve this equation requires an artifice.

Putting $x=ay$, and we have

$$a^4y^4+a^4y^3+a^4y^2+a^4y+a^4=0,$$

$$y^4+y^3+y^2+y+1=0,$$

$$y^2+\frac{1}{y^2}+y+\frac{1}{y}+1=0, \text{ putting } y+\frac{1}{y}=z, \text{ and } y^2+\frac{1}{y^2}=z^2-2,$$

we have
$$z^2-2+z+1=0,$$
$$z^2+z=1,$$
$$z=\frac{-1\pm\sqrt{5}}{2},$$
$$y+\frac{1}{y}=\frac{-1\pm\sqrt{5}}{2},$$
$$2y^2+(1\mp\sqrt{5})y=-2,$$
$$y=\frac{-1\pm\sqrt{5}\pm\sqrt{-10\mp2\sqrt{5}}}{4},$$
$$x=\frac{-a\pm a\sqrt{5}\pm a\sqrt{-10\mp2\sqrt{5}}}{4},$$
$$\therefore\quad x=a.$$
$$x=\frac{-a+a\sqrt{5}+a\sqrt{-10-2\sqrt{5}}}{4},$$
$$x=\frac{-a-a\sqrt{5}+a\sqrt{-10+2\sqrt{5}}}{4},$$
$$x=\frac{-a+a\sqrt{5}-a\sqrt{-10-2\sqrt{5}}}{4},$$
$$x=\frac{-a-a\sqrt{5}-a\sqrt{-10+2\sqrt{5}}}{4}.$$

EXAMPLES.

1. Given $x^6=1$ to find x.

Ans. $x=1,\ -1,\ \frac{1\pm\sqrt{-3}}{2}$, or $\frac{-1\pm\sqrt{-3}}{2}$.

2. Given $x^5=1$ to find x.

Ans. $x=1$, or $\frac{-1\pm\sqrt{5}\pm\sqrt{-10\mp2\sqrt{5}}}{4}$.

3. Given $x^8=a^8$ to find x.

Ans. $x=\pm a,\ \pm a\sqrt[4]{-1},\ \pm a\sqrt[4]{-1}$, or $\pm a\sqrt{-\sqrt{-1}}$.

4. Given $x^8=1$ to find x.

Ans. $x=\pm1,\ \pm\sqrt[4]{-1},\ \pm\sqrt[4]{-1}$, or $\pm\sqrt{-\sqrt{-1}}$.

5. Given $x^5+a^5=0$ to find x.

Ans. $x=-a$, or $\frac{a\pm a\sqrt{5}\pm a\sqrt{-10\pm2\sqrt{5}}}{4}$.

6. Given $x^5+1=0$ to find x.

Ans. $x=-1$, or $\frac{1\pm\sqrt{5}\pm\sqrt{-10\pm2\sqrt{-5}}}{4}$.

7. Given $x^9=a^9$ to find x.

$$Ans. \begin{cases} x=a,\ ab,\ ab^2,\ ab^3,\ ab^4,\ ab^5, \\ ab^6,\ ab^7, \text{ or } ab^8, \text{ in which} \\ b=\sqrt[3]{\dfrac{-1+\sqrt{-3}}{2}}. \end{cases}$$

8. Given $x^{10}=a^{10}$ to find x.

$$Ans. \begin{cases} x=a,\ -a, \\ x=\dfrac{-a\pm a\sqrt{5}\pm a\sqrt{-10\mp 2\sqrt{5}}}{4}. \\ x=\dfrac{a\pm a\sqrt{5}\pm a\sqrt{-10\pm 2\sqrt{5}}}{4}. \end{cases}$$

(357.) The following examples may be solved by a combination of the principles already learned. Some of them are inserted for the first time in an American work, and will be found to be the most difficult algebraic problems that have ever been published in any work upon this subject. Many of them, however, will be found to be easy of solution. Some of the values in some of the examples are omitted.

PROBLEM.

Given $2\sqrt{y^{12}+b^3y^6}-2y^6-4by^4-b^2y^2+b^3=0$ to find four values of x.

SOLUTION.

$$2\sqrt{y^{12}+b^3y^6}-2y^6-4by^4-b^2y^2+b^3=0,$$
$$2y^3\sqrt{y^6+b^3}+b^3=2y^6+4by^4+b^2y^2,$$
$$y^6+2y^3\sqrt{y^6+b^3}+y^6+b^3=4y^6+4by^4+b^2y^2,$$
$$y^3+\sqrt{y^6+b^3}=\pm(2y^3+by) \qquad (A),$$
$$\sqrt{y^6+b^3}=y^3+by,$$
$$y^6+b^3=y^6+2by^4+b^2y^2,$$
$$2by^4+b^2y^2=b^3,$$
$$2y^4+by^2=b^2,$$
$$16y^4+8by^2=8b^2,$$
$$16y^4+8by^2+b^2=9b^2,$$
$$4y^2+b=\pm 3b,$$
$$4y^2=2b, \text{ or } -4b,$$
$$2y=\pm\sqrt{2b},\ \pm 2\sqrt{-b},$$
$$y=\pm\tfrac{1}{2}\sqrt{2b},\ \pm\sqrt{-b}.$$

By taking the minus value in (A) we should obtain an equation of the sixth degree, therefore, the given equation is of the tenth degree.

EXAMPLES.

1. Given $2x^{\frac{3}{2}}(x^3+a^3)^{\frac{1}{2}}=2x^2(x+2a)+a^2(x-a)$ to find two of the five values of x. *Ans.* $x=\frac{1}{2}a$, or $-a$.

2. Given $x^6-4x^3=621$ to find all the values of x.

$$Ans.\ x=3,\ -\sqrt[3]{23},\ \frac{-3\pm 3\sqrt{-3}}{2},\ \text{or}\ \frac{\sqrt[3]{23}\pm\sqrt[3]{23}\sqrt{-3}}{2}.$$

3. Given $x^6-6x^3=16$ to find all the values of x.

$$Ans.\ x=2,\ -\sqrt[3]{2},\ \frac{-2\mp 2\sqrt{-3}}{2},\ \text{or}\ \frac{\sqrt[3]{2}\mp\sqrt[3]{2}\sqrt{-3}}{2}.$$

4. Given $x^{\frac{6}{5}}+x^{\frac{3}{5}}=756$ to find all the values of x.

$$Ans.\ x=243,\ -28\sqrt{784},\ 243\left(\frac{-1\mp\sqrt{-3}}{2}\right),\ \text{or}\ 28\sqrt{784}\left(\frac{1\mp\sqrt{-3}}{2}\right).$$

5. Given $x^3-x^{\frac{3}{2}}=56$ to find all the values of x.

$$Ans.\ x=4,\ \sqrt[3]{49},\ 1\pm\sqrt{-3},\ \text{or}\ \sqrt[3]{49}\left(\frac{-1\mp\sqrt{-3}}{2}\right).$$

6. Given $ax^{\frac{3}{2}}+bx^{\frac{3}{4}}=c$ to find two values of x.

$$Ans.\ x=\left(\frac{-b\pm\sqrt{b^2+4ac}}{2a}\right)^{\frac{4}{3}}.$$

7. Given $3x^6+42x^3=3321$ to find all the values of x.

$$Ans.\ x=3,\ -\sqrt[3]{41},\ \frac{-3\pm 3\sqrt{-3}}{2},\ \text{or}\ \frac{\sqrt[3]{41}\pm\sqrt[3]{41}\sqrt{-3}}{2}.$$

8. Given $\sqrt{x^5}-\dfrac{40}{\sqrt{x}}=3x$ to find all the values of x.

$$Ans.\ x=4,\ \sqrt[3]{25},\ -2\pm 2\sqrt{-3},\ \text{or}\ \sqrt[3]{25}\left(\frac{-1\pm\sqrt{-3}}{2}\right).$$

9. Given $(x-5)^3-3(x-5)^{\frac{3}{2}}=40$ to find all the values of x.

$$Ans.\ x=9,\ 5+\sqrt[3]{25},\ 3\pm 2\sqrt{-3},\ \text{or}\ 5+\sqrt[3]{25}\left(\frac{-1\pm\sqrt{-3}}{2}\right).$$

10. Given $\dfrac{8}{x^3}+2=\dfrac{17}{x^{\frac{3}{2}}}$ to find all the values of x.

$$Ans.\ x=4,\ \sqrt[3]{\tfrac{1}{4}},\ -2\pm 2\sqrt{-3},\ \text{or}\ \sqrt[3]{\tfrac{1}{4}}\left(\frac{-1\pm\sqrt{-3}}{2}\right).$$

11. Given $x^{\frac{7}{3}}+\frac{41\sqrt[3]{x}}{x}=\frac{97}{\sqrt[3]{x^2}}+x^{\frac{5}{6}}$ to find all the values of x.

Ans. $x=4,\ \sqrt[3]{49},\ -1\mp\sqrt{-3}$, or $\sqrt[3]{49}\left(\frac{-1\mp\sqrt{-3}}{2}\right)$.

12. Given $\frac{x^6}{2}-\frac{x^3}{4}=-\frac{1}{32}$ to find all the values of x.

Ans. $x=\sqrt[3]{\tfrac{1}{4}},\ \sqrt[3]{\tfrac{1}{4}},\ \sqrt[3]{\tfrac{1}{4}}\left(\frac{-1\pm\sqrt{-3}}{2}\right)$, or $\sqrt[3]{\tfrac{1}{4}}\left(\frac{-1\pm\sqrt{-3}}{2}\right)$.

13. Given $x^6+27x^2=2224+9x^4$ to find all the values of x.

Ans. $x=\pm 4$, or $\pm\sqrt{\frac{-7\pm 13\sqrt{-1}}{2}}$.

14. Given $(x^3+1)(x^2+1)(x+1)=30x^3$ to find all the values of x.

Ans. $x=\frac{3\pm\sqrt{5}}{2}$, or $-1\pm\frac{1}{2}\sqrt{-6}\pm\sqrt{-1\frac{1}{2}\mp\sqrt{-6}}$

15. Given $(x-\frac{1}{3})^2-\frac{25}{9}=\frac{3x^2+\frac{4}{9}}{2(x-\frac{1}{3})+\sqrt{x(x-\frac{8}{3})}}$ to find all the values of x.

Ans. $x=3,\ -\frac{1}{3},\ \frac{4\pm 2\sqrt{13}}{3}$, or $\pm\frac{2}{9}\sqrt{-3}$.

16. Given $(1-x)\sqrt{a\left(1+\frac{1}{x}\right)}-2=\sqrt{x+1}+\sqrt{3x-1}$ to find the five values of x.

Ans. $x=1$, or $\frac{-1\pm\sqrt{a+1}}{1\pm\sqrt{a-1}}$.

17. Given $3x^{\frac{4}{3}}-\frac{5x^{\frac{8}{3}}}{2}=-592$ to find the eight values of x.

Ans. $x=\pm 8,\ \pm 8\sqrt{-1},\ \pm\sqrt[4]{-(\frac{74}{5})^3}$, or $\pm\sqrt{-\sqrt{-(\frac{74}{5})^3}}$.

18. Given $\left(x^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}}+\left(a^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}}=\frac{x^2}{a}$ to find the eight values of x.

Ans. $x=\pm a\sqrt{\frac{1\pm\sqrt{5}}{2}}$, or $\pm a\sqrt{\frac{1\pm\sqrt{5}}{2}}$.

19. Given $2x\sqrt{1-x^4}=a(1+x^4)$ to find the values of x.

Ans. $x=\pm\frac{1}{a}\sqrt{-1\pm\sqrt{1-a^4}\pm\sqrt{2(1\mp\sqrt{1-a^4}}}$.

20. Given $3x^{\frac{5}{3}}+x^{\frac{5}{6}}=3104$ to find the ten values of x.

$$Ans.\ x=64,\ \left(\tfrac{97}{3}\right)^{\frac{6}{5}},\ 16\left(-1\pm\sqrt{5}\pm\sqrt{-10\mp2\sqrt{5}},\right)\ \text{or}$$

$$\left(\tfrac{97}{3}\right)^{\frac{6}{5}}\left(\frac{-1\pm\sqrt{5}\pm\sqrt{-10\mp2\sqrt{5}}}{4}\right).$$

21. Given $\dfrac{(a+x)^{\frac{1}{n}}}{a}+\dfrac{(a+x)^{\frac{1}{n}}}{x}=\dfrac{x^{\frac{1}{n}}}{c}$ to find x.

$$Ans.\ x=\frac{ac^{\frac{n}{n+1}}}{a^{\frac{n}{n+1}}-c^{\frac{n}{n+1}}}.$$

22. Given $\dfrac{mx^{\frac{m}{n}-1}}{n}=\dfrac{rx^{\frac{r}{s}-1}}{s}$ to find x. $\quad Ans.\ x=\left(\dfrac{nr}{ms}\right)^{\frac{ns}{ms-nr}}$

23. Given $x^{2n}-mx^{n}=p$ to find x. $\quad Ans.\ x=\left(\dfrac{m\pm\sqrt{m^2+4p}}{2}\right)^{\frac{1}{n}}.$

24. Given $x^{n}-2ax^{\frac{n}{2}}=b$ to find x. $\quad Ans.\ x=(a\pm\sqrt{a^2+b})^{\frac{2}{n}}.$

25. Given $3x^{2n}-2x^{n}=25$ to find x. $\quad Ans.\ x=\left(\dfrac{1+2\sqrt{19}}{3}\right)^{\frac{1}{n}}.$

26. Given $3x^{n}\sqrt[3]{x^{n}}-\dfrac{4x^{n}}{\sqrt[3]{x^{n}}}=4$ to find x.

$$Ans.\ x=(8)^{\frac{1}{2n}},\ \text{or}\ \left(-\tfrac{8}{27}\right)^{\frac{1}{2n}}.$$

27. Given $x^{4n}-2x^{3n}+x^{n}=6$ to find x.

$$Ans.\ x=\sqrt[n]{\frac{1\pm\sqrt{13}}{2}},\ \text{or}\ \sqrt[n]{\frac{1\pm\sqrt{-7}}{2}}.$$

28. Given $x^{4}-2x^{3}+x=a$ to find x.

$$Ans.\ x=\frac{1\pm\sqrt{3\pm2\sqrt{4a+1}}}{2}.$$

29. Given $a^2b^2x^{\frac{1}{n}}-4(ab)^{\frac{3}{2}}x^{\frac{m+n}{2mn}}=(a-b)^2x^{\frac{1}{m}}$ to find x.

$$Ans.\ x=\left(\frac{(\sqrt{a}+\sqrt{b})^2}{ab}\right)^{\frac{2mn}{m-n}},\ \text{or}\ \left(\frac{-(\sqrt{a}-\sqrt{b})^2}{ab}\right)^{\frac{2mn}{m-n}}.$$

30. Given $\sqrt[2pq]{x^{p+q}}=\frac{1}{2}\cdot\dfrac{a^2-b^2}{a^2+b^2}(\sqrt[p]{x}+\sqrt[q]{x})$ to find x.

$$Ans.\ x=\left(\frac{a\pm b}{a\mp b}\right)^{\frac{2pq}{p-q}}.$$

SIMULTANEOUS EQUATIONS.

PROBLEM.

(358.) Given $\left\{\begin{array}{l}\dfrac{x^2}{y^2}+\dfrac{2x+y}{\sqrt{y}}=20-\dfrac{y^2+x}{y}\\ x+8=4y\end{array}\right\}$ to find the values of x and y.

SOLUTION.

$$\frac{x^2}{y^2}+\frac{2x+y}{\sqrt{y}}=20-\frac{y^2+x}{y},$$

$$\frac{x^2}{y^2}+\frac{2x}{\sqrt{y}}+\sqrt{y}=20-y-\frac{x}{y},$$

$$\frac{x^2}{y^2}+\frac{2x}{\sqrt{y}}+y+\frac{x}{y}+\sqrt{y}=20,$$

$$\left(\frac{x}{y}+\sqrt{y}\right)^2+\left(\frac{x}{y}+\sqrt{y}\right)+\tfrac{1}{4}=20\tfrac{1}{4},$$

$$\frac{x}{y}+\sqrt{y}+\tfrac{1}{2}=\pm 4\tfrac{1}{2},$$

$$\frac{x}{y}+\sqrt{y}=4, \text{ or } -5,$$

$$x+y\sqrt{y}=4y, \text{ or } -5y,$$

$$x=4y-y\sqrt{y}, \text{ or } -5y-y\sqrt{y},$$

$x=4y-8$. Second eq. transposed.

$$\therefore\ 4y-8=4y-y\sqrt{y}, \text{ or } -5y-y\sqrt{y},$$

$y\sqrt{y}=8;$	or $y\sqrt{y}+9y=8,$
$y^{\frac{3}{2}}-8=0$	$y^{\frac{3}{2}}+9y=8,$
$(y^{\frac{1}{2}}-2)(y+2y^{\frac{1}{2}}+4)=0$	$y^{\frac{3}{2}}+1=-9y+9,$ $[y^{\frac{1}{2}}+1.$
$\therefore\ y^{\frac{1}{2}}=2$	$y^{\frac{3}{2}}+1=-9(y-1)$ dividing
$y=4$	$y-y^{\frac{1}{2}}+1=-9(y^{\frac{1}{2}}-1),$
$y+2y^{\frac{1}{2}}=-4$	$y+8y^{\frac{1}{2}}=8,$
$y^{\frac{1}{2}}=-1\pm\sqrt{-3},$	$y^{\frac{1}{2}}=-4\pm 2\sqrt{6},$
$y=-2\mp 2\sqrt{-3}$	$y=40\pm 16\sqrt{6},$
$y=-2(1\mp\sqrt{-3})$	But $y^{\frac{1}{2}}+1=0,$
	$y^{\frac{1}{2}}=-1,$
	$y=1.$

By substituting, we get $\left\{\begin{array}{l}x=8,\ -4,\ -8(2\pm\sqrt{-3}), \text{ or } 152\mp 64\sqrt{6},\\ y=4,\quad 1,\ -2(1\pm\sqrt{-3}), \text{ or } \ 40\mp 16\sqrt{6}.\end{array}\right.$

EXAMPLES.

1. Given $\left\{\begin{aligned} x^2+y^2&=\frac{13}{x-y} \\ xy&=\frac{6}{x-y}\end{aligned}\right\}$ to find the six values of x and y.

$$Ans. \left\{\begin{array}{l} x=3,\ -2,\ 3a,\ \text{or}\ -2a, \\ y=2,\ -3,\ 2a,\ \text{or}\ -3a, \\ \text{in which } a=\dfrac{-1\pm\sqrt{-3}}{2}. \end{array}\right.$$

2. Given $\left\{\begin{aligned} x^2y+xy^2&=180, \\ x^3+y^3&=189,\end{aligned}\right\}$ to find the six values of x and y.

$$Ans. \left\{\begin{array}{l} x=5,\ 4,\ 5a,\ \text{or}\ 4a, \\ y=4,\ 5,\ 4a,\ \text{or}\ 5a, \\ \text{in which } a=\dfrac{-1\pm\sqrt{-3}}{2}. \end{array}\right.$$

3. Given $\left\{\begin{array}{l} x^2y-4=4x^{\frac{1}{2}}y-\dfrac{y^3}{4}, \\ x^{\frac{3}{2}}-3=x^{\frac{1}{2}}y^{\frac{1}{2}}(x^{\frac{1}{2}}-y^{\frac{1}{2}}),\end{array}\right\}$ to find two of the six values of x and y.

$$Ans. \left\{\begin{array}{l} x=1.\ -1\mp2\sqrt{-2}, \\ y=4,\ -2. \end{array}\right.$$

4. Given $\left\{\begin{array}{l} 16x-y^{\frac{1}{2}}=6y^{\frac{1}{4}}x^{\frac{1}{2}}, \\ \dfrac{x^4}{y}-\dfrac{12}{x^2}=\dfrac{x}{\sqrt{y}},\end{array}\right\}$ to find the values of x and y.

$$Ans. \left\{\begin{array}{l} x=\pm4,\ \pm16,\ \pm2\sqrt{-3},\ \text{or}\ \pm8\sqrt{-3}, \\ y=256,\ (256)^2,\ -192,\ \text{or}\ -3(64)^2. \end{array}\right.$$

5. Given $\left\{\begin{array}{l} 2x+y=26-7\sqrt{2x+y+4}, \\ \dfrac{2x+\sqrt{y}}{2x-\sqrt{y}}=\dfrac{16}{15}+\dfrac{2x-\sqrt{y}}{2x+\sqrt{y}},\end{array}\right\}$ to find the values of x and y.

$$Ans. \left\{\begin{array}{l} x=2,\ 10,\ \dfrac{-1\mp\sqrt{321}}{64},\ 16,\ -24,\ \text{or}\ \dfrac{-1\mp\sqrt{6145}}{64}, \\[2ex] y=1,\ 25,\ \dfrac{161\pm\sqrt{321}}{32},\ 64,\ 144,\ \text{or}\ \dfrac{3073\pm\sqrt{6143}}{32}. \end{array}\right.$$

6. Given $\left\{\begin{array}{l} \dfrac{2y^2-8\sqrt{x}}{\sqrt{x}}+\sqrt{4y^2-16\sqrt{x}}=\frac{3}{2}\sqrt{x}, \\ \sqrt{x}+\sqrt{8(y-\sqrt{x})}-4=y+1. \end{array}\right\}$ to find the values of x and y.

$$Ans. \left\{\begin{array}{l} x=4,\ \frac{4}{9}\dfrac{-4\mp16\sqrt{-39}}{3},\ \frac{4}{25},\ 4, \text{ or } \dfrac{788\pm24\sqrt{644}}{25}, \\ y=3,\ 1\frac{2}{3}\ \dfrac{3\pm2\sqrt{-39}}{3},\ 1\frac{2}{5},\ -1, \text{ or } \dfrac{37\pm\sqrt{644}}{5}. \end{array}\right.$$

7. Given $\left\{\begin{array}{l} x^4-2x^2y+y^2=49, \\ x^4-2x^2y^2+y^4-x^2+y^2=20, \end{array}\right\}$ to find the sixteen values of x and y.

$$Ans. \left\{\begin{array}{l} x=\pm3,\ \pm\sqrt{6},\ \pm\sqrt{\dfrac{-13\pm\sqrt{-47}}{2}},\ \pm\sqrt{\dfrac{15\pm3\sqrt{5}}{2}}, \text{ or} \\ \qquad\qquad \pm\sqrt{\dfrac{-13\pm\sqrt{-11}}{2}}, \\ y=2,\ -1,\ \dfrac{1\pm\sqrt{-47}}{2},\ \dfrac{1\pm3\sqrt{5}}{2}, \text{ or } \dfrac{1\pm\sqrt{-11}}{2}. \end{array}\right.$$

8. Given $\left\{\begin{array}{l} x+y+xy+x^2y+xy^2+x^3y+2x^2y^2+xy^3+x^3y^2 \\ \qquad +x^2y^3=11, \\ x^4y+3x^3y^2+3x^2y^3+2x^4y^2+4x^3y^3+2x^2y^4 \\ \qquad +4x^4y^3+4x^3y^4+xy^4+x^5y^2+x^5y^3+2x^4y^4 \\ \qquad +x^2y^5+x^3y^5=30 \end{array}\right\}$

to find the sixteen values of x and y.

$$Ans. \left\{\begin{array}{l} x+y=\frac{5}{2}\pm\frac{1}{2}\sqrt{21},\ \frac{1}{2}\pm\frac{1}{2}\sqrt{-19},\ 2 \text{ or } 1, \text{ or } 1\ \pm\sqrt{-2}, \\ xy=\frac{5}{2}\mp\frac{1}{2}\sqrt{21},\ \frac{1}{2}\mp\frac{1}{2}\sqrt{-19},\ 1 \text{ or } 2, \text{ or } 1\ \mp\sqrt{-2}. \end{array}\right.$$

NOTE.—The solution of these eight simultaneous equations will give the sixteen values of x and y.

9. Given $x^2y\sqrt{xy}=a$, $xz^2\sqrt{xz}=b$, $y^2z\sqrt{zy}=c$ to find x, y, and z.

$$Ans.\ x=\left(\frac{a^4\sqrt[4]{(abc)^9}}{c^6}\right)^{\frac{1}{19}},\ y=\left(\frac{c^4\sqrt[4]{(abc)^9}}{b^6}\right)^{\frac{1}{19}},\ z=\left(\frac{b^4\sqrt[4]{(abc)^9}}{a^6}\right)^{\frac{1}{19}}.$$

10. Given $\left\{\begin{array}{l} x^n+y^n=2a^n \\ xy=c^2 \end{array}\right\}$ to find the values of x and y.

$$Ans. \left\{\begin{array}{l} x=(a^n\pm\sqrt{a^{2n}-c^{2n}})^{\frac{1}{n}}, \\ y=\dfrac{c^2}{(a^n\pm\sqrt{a^{2n}-c^{2n}})^{\frac{1}{n}}}. \end{array}\right.$$

CHAPTER XV.

ARITHMETICAL PROGRESSION.

(359.) AN *Arithmetical Progression* is a series of quantities in which the difference of the consecutive quantities is constant, as

$$\div a \cdot a \pm d \cdot a \pm 2d \cdot a \pm 3d \cdot a \pm 4d., \text{ \&c.}$$

PROBLEM.

(360.) *To find a general expression for any term of an arithmetical progression.*

SOLUTION.

In the progression $\overset{1st,}{a}, \overset{2d,}{a \pm d}, \overset{3d,}{a \pm 2d}, \overset{4th,}{a \pm 3d}, \overset{5th,}{a \pm 4d}, \overset{6th,}{a \pm 5d}$, &c., we see that any term is equal to a *plus*, or *minus* d, affected by a coefficient which is one less than the number of the term; therefore, if we let n represent the number of any term, we have the general expression

$$n\text{th term} = a \pm (n-1)d.$$

If we suppose the progression to terminate, we have

$$l = a \pm (n-1)d,$$

in which l represents the last term, n the whole number of terms, d the common difference, and a the first term.

PROBLEM.

(361.) *To find a general expression for the sum of all the terms of an arithmetical progression.*

SOLUTION.

Putting S equal to the sum of all the terms in a progression, we have

$$S = a + a \pm d + a \pm 2d + a \pm 3d \,\ldots\ldots\ldots\, l, \text{ to } n \text{ terms, (1).}$$

or, $S = l + l \mp d + l \mp 2d + l \mp 3d \,\ldots\ldots\ldots\, a,$ " " (2).

$$2S = (a+l) + (a+l) + (a+l) + (a+l) \,\ldots\ldots\, (l+a) \quad (3) = (1) + (2).$$

Since, $2S$ equals $(a+l)$ taken n times, the expression becomes

$$2S=(a+l)n,$$

$$S=\left(\frac{a+l}{2}\right)n \text{ or } (a+l)\frac{n}{2},$$

which is the expression required.

REMARK.—By the aid of the two formulas $l=a\pm(n-1)d$, and $S=\left(\frac{a+l}{2}\right)n$, we are enabled to find any two of the terms a, d, n, l, S, when three of them are given since we shall have two equations and two unknown quantities. We append a few simple propositions for the student to demonstrate.

PROPOSITION

1. *In an arithmetical progression consisting of three terms, the sum of the first and the third term is twice the second.*

PROPOSITION

2. *In an arithmetical progression consisting of four terms, the sum of the first and the fourth is equal to the sum of the second and the third.*

PROPOSITION

3. *In an arithmetical progression consisting of any number of terms, the sum of any two terms equally distant from the extremes is equal to the sum of the extremes.*

PROPOSITION

4. *In an arithmetical progression consisting of an odd number of terms, twice the middle term is equal to the sum of the extremes.*

REMARK.—In the following examples the known terms should be substituted instead of the letters representing them, and there will thus arise one or two equations, according as one or both of the formulæ, $L=a\pm(n-1)d$ and $S=\left(\frac{a+l}{2}\right)n$, are involved.

QUESTIONS.

1. The first term of an arithmetical progression is 2, the common difference 3, and the number of terms 8. What is the last term?

Ans. 23.

2. The first term of an arithmetical progression is 3, the common difference 2, and the last term 99. What is the number of terms?

Ans. 49.

3. The last term of an arithmetical progression is 100, the common difference 4, and the number of terms 30. What is the first term?

Ans. -16.

4. The first term of an arithmetical progression is -20, the number of terms 51, and the last term 230. What is the common difference?

Ans. 5.

5. The first term of an arithmetical progression is -12, the common difference -7, and the number of terms 101. What is the sum of the series?

Ans. -36057.

6. Insert 8 arithmetical means between 3 and 21.

Ans. $\div 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot 15 \cdot 17 \cdot 19$.

7. Insert 3 arithmetical means between $\frac{1}{3}$ and $\frac{1}{2}$.

Ans. $\div \frac{3}{8} \cdot \frac{5}{12} \cdot \frac{11}{24}$.

8. The sum of an arithmetical series is 108, the first term 3, and the common difference 2. What is the number of terms?

Ans. 10.

9. What is the sum of n terms of the progression $\div 1 \cdot 2 \cdot 3 \cdot 4 \cdot$ &c.?

Ans. $S=\left(\frac{1+n}{2}\right)n$.

10. The first term of an arithmetical progression is 14, and the sum of eight terms is 28. What is the common difference?

Ans. -3.

11. The first term of an arithmetical progression is 12, and the common difference $-\frac{1}{2}$. What is the sum of the series, supposing all its terms to be positive?

Ans. 150.

12. What is the sum of the series $\div 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot$ to 100 terms?

Ans. 10000.

13. The first term of an arithmetical progression is $\frac{1}{2}$, the common difference $\frac{1}{2}$, and the number of terms 25. What is the sum of the series?

Ans. $162\frac{1}{2}$.

14. The first term of an arithmetical progression is 1, the number of terms 23, and the sum of the series is $149\frac{1}{2}$. What is the common difference?

Ans. $\frac{1}{2}$.

15. What is the nth term of the series $\div 1 \cdot 3 \cdot 5 \cdot 7 \cdot$ &c.?

Ans. $2n-1$.

16. What is the sum of n terms of the series $\div 1 \cdot 3 \cdot 5 \cdot 7 \cdot$ &c.?

Ans. n^2.

17. If a body falling to the earth descends a feet the first second, $3a$ the second, $5a$ the third, and so on. How far will it fall during the tth second? *Ans.* $(2t-1)a$.

18. If a body falling to the earth descends a feet the first second, $3a$ the second, $5a$ the third, and so on. How far will it fall in t seconds? *Ans.* mt^2.

19. The first term of an arithmetical progression is $\frac{5}{7}$, the common difference is $1\frac{2}{3}$, and the number of terms 13. What is the sum of the series? *Ans.* $139\frac{2}{7}$.

20. The first term of an arithmetical progression is $-\frac{3}{4}$, the common difference $-\frac{7}{8}$, and the number of terms 25. What is the sum of the series? *Ans.* $-281\frac{1}{4}$.

(362.) There are problems in arithmetical progression to which the fundamental formulæ are not immediately applicable, since three of the quantities a, d, n, l, s are not given to find the other two, but, in every case, the number of terms being given together with other conditions to find the terms.

It is necessary for the student to know how to represent in the best manner a series of numbers in arithmetical progression. One mode has already been given, namely, $\div x \cdot x \pm y \cdot x \pm 2y \cdot x \pm 3y \cdot$ &c., in which x represents the first term and y the common difference. This mode of representation, however, is seldom the most expedient.

When the number of terms is odd, assume the middle one to be equal to x, and y the common difference; thus,

$\div x-y \cdot x \cdot x+y \cdot$ when there are three terms.

$\div x-2y \cdot x-y \cdot x \cdot x+y \cdot x+2y$ " " five "

When the terms are even, put $x-y$ and $x+y$ equal to the middle terms, $2y$ being equal to the common difference; thus,

$\div x-3y \cdot x-y \cdot x+y \cdot x+3y$ when there are four terms.

$\div x-5y \cdot x-3y \cdot x-y \cdot x+y \cdot x+3y \cdot x+5y \cdot$ " " six "

It may be seen that in this mode of representation the common difference disappears. The formula for the sum of the series may also be easily deduced from this method of representation.

Thus, $\div \ldots\ldots x-2y \cdot x-y \cdot x \cdot x+y \cdot x+2y \ldots\ldots$ to n terms.

and thus, $\div \ldots\ldots x-3y \cdot x-y \cdot x+y \cdot x+3y \ldots\ldots$ to n terms.

It is evident that in either of these cases the sum of the series is n times x, or nx. But x may be considered equal to half the sum of the first and last term, or $=\frac{a+l}{2}$, therefore $S=n\,x=\left(\frac{a+l}{2}\right)n$.

QUESTION.

What four numbers are there in arithmetical progression, of which the sum of the squares of the extremes is 200, and the sum of the squares of the means is 136?

SOLUTION.

Let $x+3y$, $x+y$, $x-y$, $x-3y$ represent the numbers:

$$\begin{aligned} \text{then}\quad 2x^2+18y^2&=200,\\ \text{and}\quad 2x^2+2y^2&=136,\\ \therefore\quad 16y^2&=64,\\ 4y&=\pm 8,\\ y&=\pm 2,\\ \text{whence,}\quad 2x^2=136-2y^2&=128,\\ x&=\pm 8, \end{aligned}$$

$\therefore$ the numbers are ± 14, ± 10, ± 6, and ± 2.

QUESTIONS.

1. Four numbers are in arithmetical progression. The sum of their squares is equal to 276, and the sum of the numbers themselves is 32. What are the numbers? *Ans.* 11, 9, 7, and 5.

2. A number consists of 3 digits, which are in arithmetical progression; and this number divided by the sum of its digits is equal to 26; but if 198 be added to it, the digits will be inverted. What is the number? *Ans.* 234.

3. The sum of four integral numbers in arithmetical progression is 20, and the sum of their reciprocals is $\frac{25}{24}$. What are the numbers? *Ans.* 2, 4, 6, and 8.

4. The sum of \$27 was to be raised by subscription by three persons, A, B, and C; the sums to be subscribed by them respectively forming an arithmetical progression. But C, dying before the money was paid, the whole fell to A and B; and C's share was raised between them in the proportion of $3:2$, when it appeared that the whole sum subscribed by A was to the whole sum subscribed by $B :: 4 : 5$. What were the original subscriptions of A, B, and C?

Ans. A's \$3, B's \$9, and C's \$15.

5. After A, who went at the rate of 4 miles an hour, had traveled $2\frac{3}{4}$ hours, B set out to overtake him, and in order thereto went $4\frac{1}{2}$

miles the first hour, $4\frac{3}{4}$ the second, 5 the third, and so on, gaining $\frac{1}{4}$ of a mile every hour. In how many hours would he overtake A?

Ans. 8 hours.

6. The base of a right-angled triangle is 6, and the sides are in arithmetical progression. What are the other two sides?

Ans. 6, 8, and 10, or $4\frac{1}{2}$, 6, and $7\frac{1}{2}$.

7. A and B set out from London at the same time, to go round the world (23661 miles); one going east, the other west. A goes 1 mile the first day, 2 the second, and so on. B goes 20 miles a day. In how many days will they meet, and how many miles will each have traveled?

Ans. 198 days. A goes 19701, and B 3960 miles.

8. A traveler sets out for a certain place, and travels 1 mile the first day, 2 the second, and so on. In 5 days afterwards another sets out, and travels 12 miles a day. How long must the second travel to overtake the first?

Ans. 3, or 10 days. Explain this result.

9. A and B, 165 miles distant from each other, set out with a design to meet; A travels 1 mile the first day, 2 the second, 3 the third, and so on; B travels 20 miles the first day, 18 the second, 16 the third, and so on. How soon will they be together?

Ans. In 10 or 33 days. Explain the last result.

10. There are four numbers in arithmetical progression whose continued product is 1680, and common difference 4. What are the numbers? *Ans.* ± 14, ± 10, ± 6, and ± 2.

11. The product of five numbers in arithmetical progression is 945, and their sum is 25. What are the numbers? *Ans.* 9, 7, 5, 3, and 1.

12. There are three numbers in arithmetical progression, and the square of the first added to the product of the other two is 16; the square of the second added to the product of the other two is 14. What are the numbers?

Ans. 1, 3, and 5; or -5, -3, and -1.

13. There are two casks, A and B, of which, A the greater, holds 312 gallons. Into A a certain quantity of wine is put, and B is filled with water; then water is conveyed out of B into A in the following manner. First, a number of gallons is taken, which is less by 2 than the square root of the number of gallons in A; then a quantity less

than the former by 2 gallons, and so on. Now when B is in this manner exactly emptied, A is exactly full; and it is known that 8 gallons were taken out of B at one time, after which the quantity left in B was 12 gallons. What is the number of gallons of wine in A?

Ans. 256.

14. From two towns which were 168 miles distant, two persons, A and B, set out to meet each other; A went 3 miles the first day, 5 the next, 7 the third, and so on; B went 4 miles the first day, 6 the next, and so on. In how many days did they meet? *Ans.* 8.

15. A man borrowed \$60; what sum shall he pay daily, to cancel the debt, principal and interest, in 60 days; interest at 10 per cent. for 12 months, of 30 days each? *Ans.* \$1 and $\frac{9}{7}\frac{1}{2}$ of a cent.

CHAPTER XVI.

GEOMETRICAL PROGRESSION.

(363.) A GEOMETRICAL PROGRESSION is a series in which the successive quantities are formed by multiplying the preceding one by a constant quantity, which is called the ratio of the progression; as,

$$\div\!\!\cdot\; 2 : 4 : 8 : 16 : 32 : 64 \text{ in which the ratio is } 2.$$
$$\div\!\!\cdot\; 27 : 9 : 3 : 1 : \tfrac{1}{3} : \tfrac{1}{9} \text{ in which the ratio is } \tfrac{1}{3}.$$
$$\div\!\!\cdot\; a : ar : ar^2 : ar^3 : ar^4 \text{ in which the ratio is } r.$$

REMARK.—The *ratio* of a *geometrical* progression is the constant multiplier, and it would be more philosophical to use another term, as the French do. We suggest the word *rate.* Some English writers say that the *ratio* of a *geometrical* progression is the *inverse* ratio of its consecutive terms. *Briot,* a French writer, says, that "*The* (RAPPORT), RATIO *of each term to the preceding is called the* (RAISON), RATE." A few say that the direct ratio of two consecutive numbers is equal to the second divided by the first. This is not only unphilosophical but is not consistent with the symbol used to express *ratio.* Thus, $a : b = c : d$ is read the ratio of a to b equals the ratio of c to d. Now, the symbol : is a sign of division, and is generally used as such by the Germans in preference to the symbol $\div$ introduced by Dr. Pell. Several American writers erroneously call the method of dividing *consequent* by *antecedent* to express the ratio of the latter to the former, the *French method.* Lacroix is the only French author that we have noticed who has adopted this plan.

PROBLEM.

(364.) *To find an expression for the* nth *term of a geometrical progression.*

SOLUTION.

In the series $\div\!\!\cdot\; \overset{1st,}{a} : \overset{2d,}{ar} : \overset{3d,}{ar^2} : \overset{4th,}{ar^3} : \overset{5th,}{ar^4} : \overset{6th,}{ar^5} :$ &c., it may be seen that any term is equal to the first multiplied by the ratio affected by an exponent which is one less than the number of the term.

Therefore, the nth term $= ar^{n-1}$.

In a series which terminates, if we represent the number of terms by n, and the last term by l, we have

$$l=ar^{n-1}.$$

PROBLEM.

(365.) To find an expression for the sum of the terms of a geometrical progression.

SOLUTION.

Representing the sum by S, we have

$$S=a+ar+ar^2+ar^3+ar^4 \ldots\ldots\ldots ar^{n-3}+ar^{n-2}+ar^{n-1}, \quad (1).$$

$$rS=ar+ar^2+ar^3+ar^4+ar^5 \ldots\ldots\ldots ar^{n-2}+ar^{n-1}+ar^n \quad (2)=(1)\times r.$$

then $rS-S=ar^n-a$ $\quad$ (3)=(2)-(1).

Since $ar^n=ar^{n-1}\times r$, and $ar^{n-1}=l$, we have $ar^n=lr$;

$$\therefore \quad rS-S=ar^n-a,$$

$$\text{becomes } rS-S=lr-a$$

$$(r-1)S=lr-a$$

$$S=\frac{lr-a}{r-1}$$

which is the expression required. When r is less than 1 it is best to put $S=\frac{a-lr}{1-r}$, although the same result will be obtained from both forms.

PROBLEM.

(366.) To find an expression for the sum of the terms of a decreasing geometrical progression when the number of terms is infinite.

SOLUTION.

It may be seen from the formula $S=\frac{a-lr}{1-r}$ that the sum of the series depends upon the first term, last term, and ratio. In a decreasing geometrical series the terms must continually approach zero as a limit. Therefore, when the number of terms is infinite, we are compelled to consider zero as the last term; since there is no quantity, however small, greater than zero that may not be reached or passed by a finite number of terms. If, then, $l=0$, lr must also $=0$, and the above formula becomes, for a decreasing geometrical progression having an infinite number of terms, $S=\frac{a}{1-r}$, which is the basis of the following

RULE.

Divide the first term of an infinitely decreasing geometrical progression by the difference between unity and the ratio, and the result will be the sum of the series.

The above formula may also be deduced in the following manner:

Let $S=a+ar+ar^2+ar^3$, &c., *ad infinitum*, r being less than 1 (1)

$rS=\quad +ar+ar^2+ar^3$, &c., " " " " (2)=(1)×r

$rS-S=a$ (3)=(2)−(1)

$$\therefore S=\frac{a}{r-1}, \text{ the same as before.}$$

A few simple propositions are here appended for the student to demonstrate.

PROPOSITION

(367.) 1. *In a geometrical progression consisting of three terms the product of the extremes is equal to the square of the mean.*

PROPOSITION

(368.) 2. *In a geometrical progression consisting of four terms the product of the extremes is equal to the product of the means.*

PROPOSITION

(369.) 3. *In a geometrical progression consisting of any number of terms the product of the extremes is equal to the product of any two terms equally distant from them.*

EXAMPLES.

1. Find the 11th term of $\div\!\!\div 3 : 6 : 12 :$ &c. *Ans.* 3072.

2. Find the sum of 9 terms of $\div\!\!\div 1 : 2 : 4 :$ &c. *Ans.* 511.

3. Find the ratio when the first term is 3, last term 768, and number of terms 9. *Ans.* 2.

4. Find the 11th term of $\div\!\!\div \frac{5}{6} : \frac{5}{9} : \frac{10}{27} :$ &c. *Ans.* $\frac{2560}{177147}$.

5. Find three geometric means between 4 and 324. *Ans.* 12, 36, and 108.

6. Find three geometric means between $\frac{1}{2}$ and $\frac{2}{9}$. *Ans.* $\frac{1}{6}\sqrt{6}$, $\frac{1}{3}$, and $\frac{1}{9}\sqrt{6}$.

7. Find the sum of $\div\!\!\div 1 : \frac{1}{2} : \frac{1}{4} :$ &c. to infinity. *Ans.* 2.

8. Find the value of .3333, &c., or $\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}$, &c., to infinity. *Ans.* $\frac{1}{3}$.

9. Find the value of 9.99999, &c., or $9+\frac{9}{10}+\frac{9}{100}$, &c., to infinity. *Ans.* 10.

10. Find the value of $\div\!\!\div \frac{3}{5} : -\frac{6}{25} : \frac{12}{125} : -\frac{24}{625}$: &c. to infinity. *Ans.* $\frac{3}{7}$.

11. Find the sum of $\div\!\!\div 1 : \frac{1}{x} : \frac{1}{x^2}$: &c., to infinity. *Ans.* $\frac{x}{x-1}$.

12. Find the value of .2333, &c., to infinity. *Ans.* $\frac{7}{30}$.

13. Find the value of .3411111, &c., to infinity. *Ans.* $\frac{307}{900}$.

14. Find the value of .323232, &c., to infinity. *Ans.* $\frac{32}{99}$.

15. Find the value of .20414141, &c., to infinity. *Ans.* $\frac{2021}{9900}$.

16. Find the sum of the series 40, 16, &c., to infinity. *Ans.* $66\frac{2}{3}$.

17. Find the sum of $\div\!\!\div x^3 : ax : \frac{a^2}{x}$: &c., to infinity. *Ans.* $\frac{x^5}{x^2-a}$.

18. Find the sum of $\div\!\!\div x^{\frac{3}{4}} : -\frac{a}{x^2}$, &c., to infinity. *Ans.* $\frac{x^{\frac{7}{2}}}{x^{\frac{11}{4}}+a}$.

19. Suppose a body to move eternally in this manner, viz., 20 miles the first minute, 19 miles the second minute, $18\frac{1}{20}$ the third, and so on in geometrical progression. What is the utmost distance it can reach? *Ans.* 400 miles.

20. What is the distance passed through by a ball, before it comes to rest, which falls from the height of 50 feet, and at every fall rebounds half the distance? *Ans.* 150.

21. In the preceding problem, supposing that a body falls $16\frac{1}{2}$ feet the first second, 3 times as far the next second, and 5 times as far the third second, and so on, how long will it be before it comes to rest?

Ans. $\frac{10}{193}\sqrt{579}(4+3\sqrt{2})=10{\cdot}2785222$ seconds.

(370.) There are many interesting problems in geometrical progression to which the fundamental formulæ do not immediately apply. In their solution a great deal frequently depends upon the notation used.

$\div\!\!\div\, x : xy : xy^2$ is a progression of three terms, y being the ratio.

$\div\!\!\div\, x : \sqrt{xy} : y$ is a progression of three terms, $\sqrt{\frac{y}{x}}$ being the ratio.

This last method, however, may be best explained by the principle in Prop. 1, (**367.**)

$\because x : xy : xy^2 : xy^3$ is a progression of four terms, y being the ratio.

$\because \frac{x^2}{y} : x : y : \frac{y^2}{x}$ is a progression of four terms, $\frac{y}{x}$ being the ratio.

$\because x : xy : xy^2 : xy^3 : xy^4$ is a progression of five terms, y being the ratio.

$\because \frac{x^3}{y} : x^2 : xy : y^2 : \frac{y^3}{x}$ is a progression of five terms, $\frac{y}{x}$ being the ratio.

$\because x : xy : xy^2 : xy^3 : xy^4 : xy^5$ is a progression of six terms, y being the ratio.

$\because \frac{x^3}{y^2} : \frac{x^2}{y} : x : y : \frac{y^2}{x} \cdot \frac{y^3}{x^2}$ is a progression of six terms, $\frac{y}{x}$ being the ratio.

(**371.**) It is sometimes expedient to employ substitution in the solution of geometrical problems. For example, if we put

$$x+y=s,$$

and $$xy=p,$$

we get $$x^2+y^2=s^2-2p,$$

and $$x^3+y^3=s^3-3ps,$$

and $$x^4+y^4=s^4-4s^2p+2p^2,$$

and $$x^5+y^5=s^5-5s^3p+5sp^2.$$

REMARK.—It would be a good exercise for the student to ascertain how these results are obtained.

QUESTION.

(**372.**) What six numbers in geometrical progression are those of which the sum of the extremes is 99 and the sum of the other four terms 90?

SOLUTION.

The conditions show that the sum of the six numbers is 189.

Let x, xy, xy^2, xy^3, xy^4, xy^5, represent the numbers.

The formula $S=\frac{lr-a}{r-1}$ becomes by substitution

$$189=\frac{xy^6-x}{y-1}=\frac{x(y^6-1)}{y-1},$$

$$x=\frac{189(y-1)}{y^6-1},$$ [forward

$$\text{But}\quad xy^5+x=99,$$

$$\therefore\quad x=\frac{99}{y^5+1},$$

$$\therefore\quad \frac{189(y-1)}{y^6-1}=\frac{99}{y^5+1},$$

$$\frac{21}{y^4+y^2+1}=\frac{11}{y^4-y^3+y^2-y+1},$$

$$21y^4-21y^3+21y^2-21y+21=11y^4+11y^2+11,$$

$$10y^4+10y^2+10=21y^3+21y \qquad (C),$$

$$10(y^4+y^2+1)=21y(y^2+1),$$

Putting $y^2+1=z$, we have $10(z^2-y^2)=21yz$,

$$10z^2-21yz=10y^2,$$

$$z=\frac{21y\pm29y}{20}=\frac{5y}{2},$$

$$y^2+1=\frac{5y}{2},$$

$$2y^2-5y=-2,$$

$$y=\frac{5\pm3}{4}=2,$$

$$x=\frac{99}{y^5+1}=\frac{99}{33}=3,$$

Therefore, the progression is ∺3 : 6 : 12 : 24 : 48 : 96.

NOTE.—Equation (C) is recurring, and might be solved according to either of the methods given in biquadratics. Equation (C) might have been obtained without using the general formula for the sum of the series.

ANOTHER SOLUTION.

Let $\frac{x^3}{y^2}, \frac{x^2}{y}, x, y, \frac{y^2}{x}, \frac{y^3}{x^2}$ represent the numbers.

$$\therefore\quad \frac{x^3}{y^2}+\frac{y^3}{x^2}=99 \qquad (1),$$

$$\text{and}\quad \frac{x^2}{y}+x+y+\frac{y^2}{x}=90 \qquad (2),$$

$x^5+y^5=99x^2y^2$ (3)$=$(1)$\times x^2y^2$. Putting $x+y=s$ and $xy=p$, we have $s^5-5s^3p+5sp^2=99p^2$ (4), $\times xy$

$x^3+xy(x+y)+y^3=90xy$ (5)$=$(2),

$x^3+y^3=90xy-xy(x+y)$ (6)$=$(5) transposed.

$s^3-3sp=90p-sp,$ [forward

$$s^3=90p+2sp,$$

$$p=\frac{s^3}{90+2s} \quad (9),$$

$$s^6-\frac{5s^6}{90+2s}+\frac{5s}{(90+2s)^2}=\frac{99s^6}{(90+2s)^2} \quad (10)=(9) \text{ substituted in } (4).$$

$$1-\frac{5s}{90+2s}+\frac{5s^2}{(90+2s)^2}=\frac{99s}{(90+2s)^2},$$

$$8100+360s+4s^2-450s-10s^2+5s^2=99s,$$

$$s^2+189s=8100,$$

$$s=\frac{-189\pm261}{2}=36,$$

$$p=\frac{s^3}{90+2s}=\frac{36\cdot36\cdot36}{2\cdot9\cdot9}=18\cdot4\cdot4=12\cdot24,$$

$$x+y=36,$$

$xy=12\cdot24.$ Whence, it is obvious without solution, that $x=12,$

and $y=24.$

Therefore, the series is $\div\!\div 3:6:12:24:48:96.$

REMARK.—This problem is one of the most difficult of those generally proposed in geometrical progression, and the solutions given should be carefully studied by the student that he may be able to solve others of like character.

QUESTIONS.

1. The sum of the first and third of four numbers in geometrical progression is 148, and the sum of the second and fourth is 888. What are the numbers? *Ans.* 4, 24, 144, and 864.

2. There are three numbers in geometrical progression, the sum of the first and second is 15, and the differences of the second and third is 36. What are the numbers? *Ans.* 3, 12, and 48.

3. What three numbers are there in geometrical progression whose sum is 14, and the sum of whose squares is 84? *Ans.* 2, 4, and 8.

4. What three numbers are those in geometrical progression, whose sum is 52, and the sum of whose extremes is to the mean as 10 to 3? *Ans.* 4, 12, and 36.

5. What three numbers are those in geometrical progression, whose sum is 13, and the sum of whose extremes multiplied by the mean is 30? *Ans.* 1, 3, and 9.

6. The sum of the first and second of four numbers in geometrical

progression is 15, and the sum of the third and fourth is 60. What are the numbers?

Ans. 5, 10, 20, and 40; or -15, 30, -60, and 120.

7. The sum of four numbers in geometrical progression is equal to the common ratio $+1$; and the first term $=\frac{1}{17}$. What are the numbers? *Ans.* $\frac{1}{17}$, $\frac{4}{17}$, $\frac{16}{17}$, and $\frac{64}{17}$.

8. A gentleman divided $210 among three servants; the sums received were in geometrical progression, and the first received $90 more than the last. How many dollars did each receive?

Ans. $120, $60, and $30.

9. The sum of three numbers in geometrical progression is 35, and the mean term is to the difference of the extremes as 2 to 3. What are the numbers? *Ans.* 5, 10, and 20.

10. There are three numbers in geometrical progression, the greatest of which exceeds the least by 15. Also, the difference of the squares of the greatest and the least, is to the sum of the squares of all the three numbers as 5 : 7. What are the numbers?

Ans. 5, 10, and 20.

11. The sum of three numbers in geometrical progression is 13, and the product of the mean and sum of the extremes is 30. What are the numbers? *Ans.* 1, 3, and 9.

12. The diagonals of 4 squares are in an increasing geometrical progression, and the product of the squares of the diagonals of the extremes is to the product of the diagonals of the means as a side of the third is to the square root of the common ratio divided by $4\sqrt{2}$. What is the diagonal of the third square, and the common ratio, supposing their difference equal to 45?

Ans. 81 the ratio, and 36 the diagonal of the 3d square.

13. The difference between the first and second of four numbers in geometrical progression is 36, and the difference between the third and fourth is 4. What are the numbers? *Ans.* 54, 18, 6, and 2.

14. There are three numbers in geometrical progression, the sum of the first and second of which is 9, and the sum of the first and third is 15. What are the numbers? *Ans.* 3, 6, and 12.

15. There are three numbers in geometrical progression, whose sum is 14; and the sum of the first and second is to the sum of the second and third as 1 to 2. What are the numbers?

Ans. 2, 4, and 8.

16. There are three numbers in geometrical progression, whose continued product is 64, and the sum of their cubes is 584. What are the numbers? *Ans.* 2, 4, and 8.

17. There are four numbers in geometrical progression, the second of which is less than the fourth by 24; and the sum of the extremes is to the sum of the means as 7 to 3. What are the numbers?

Ans. 1, 3, 9, and 27.

18. The sum of $700 was divided among four persons, whose shares were in geometrical progression; and the difference between the greatest and least was to the difference between the means as 37 to 12. What were their respective shares?

Ans. $108, $144, $192, and $256.

19. A company of merchants fitted out a privateer, each merchant subscribing $100. The captain subscribed nothing, but was entitled to a $100 share, at the end of every certain number of months. In the course of 25 months he captured three prizes, which were in geometrical progression, the middle term being $\frac{1}{4}$ the cost of the equipment, the common ratio the number of months which entitled the captain to his $100 share, and their sum $1375 more than the cost of the equipment. After deducting $875 for prize money to the crew, the captain's share of the remainder amounted to $\frac{1}{5}$ of that of the company. What was the number of merchants, and the captain's pay?

Ans. 25 merchants, and captain's pay $100 at the end of every 5 months.

20. There are four numbers in geometrical progression, the difference of whose means is 3, and the difference of whose extremes is $10\frac{1}{2}$. What are the numbers? *Ans.* $1\frac{1}{2}$, 3, 6, and 12.

21. The sum of three numbers in geometrical progression is 31, and the sum of their square is 651. What are the numbers?

Ans. 1, 5, and 25.

22. The sum of four numbers in geometrical progression is 15, and the sum of their squares is 85. What are the numbers?

Ans. 1, 2, 4, and 8.

23. The sum of five numbers in geometrical progression is 31, and the sum of their squares is 341. What are the numbers?

Ans. 1, 2, 4, 8, and 16.

24. The sum of six numbers in geometrical progression is 94, and the sum of the second and fifth is 27. What are the numbers?

Ans. 1, 3, 6, 12, 24, and 48.

25. The sum of six numbers in geometrical progression is 63, and the sum of the means is 12. What are the numbers?

Ans. 1, 2, 4, 8, 16, and 32.

26. The sum of six numbers in geometrical progression is 1365, and the sum of the extremes is 1025. What are the numbers?

Ans. 1, 4, 16, 64, 256, and 1024.

27. What six numbers are those in geometrical progression whose sum is 315, and the sum of whose extremes is 165?

Ans. 5, 10, 20, 40, 80, and 160.

28. What number is that which being severally added to 3, 19, and 51, shall make the results in geometrical progression?

Ans. 13.

29. $120 are divided between four persons, in such a way, that their shares may be in arithmetical progression; but if the second and third had received $12 less each, and the fourth $24 more, the shares would have been in geometrical progression. What was the share of each? *Ans.* $3, $21, $39, and $57 respectively.

30. The sum of three numbers in geometrical progression is 7, and the difference of whose difference is 1. What are the numbers?

Ans. 1, 2, and 4.

CHAPTER XVII.

PROPORTION.

(373.) PROPORTION is an equality of ratios.

(374.) If the ratio of a to b is equal to the ratio of c to d, these four terms constitute a proportion which is usually written $a : b :: c : d$, and is read a is to b as c is to d. Sometimes the sign of equality is used instead of the four dots, as $a : b = c : d$, which may be read, the ratio of a to b is equal to the ratio of c to d.

We may consider the symbol : as an abbreviation of the sign $\div$; whence, we infer that $a : b :: c : d$ is only another mode of writing $a \div b = c \div d$, which is the same as $\frac{a}{b} = \frac{c}{d}$. This shows that every proportion is essentially an equation.

(375.) The four quantities of a proportion are called its *terms*.

(376.) The first and the fourth term are called the *extremes*, and the second and the third term, the *means*.

(377.) The first two terms of a proportion are the *first couplet*, and the other two, the *second couplet*.

(378.) The first term of a couplet is called the *antecedent*, and the second term the *consequent*.

(379.) *Three quantities* are in proportion when the ratio of the first to the second is equal to the ratio of the second to the third.

(380.) The second quantity is called a *mean proportional* between the other two, and the third quantity a *third* proportional to the other two.

Thus, in the proportion $a : b :: b : c$, b is the *mean proportional*, and c the *third proportional*.

(381.) The equality of more than two ratios may be thus written, $a : b :: c : d :: e : f :: g : h$, &c., which may be read a is to b as c is to d, as e is to f, as g is to h, &c.

REMARK.—The student should observe that the demonstrations of the following propositions in regard to proportion are based upon the fact that every proportion is essentially an equation.

PROPOSITION

(382.) 1. *In every proportion the product of the extremes is equal to the product of the means.*

DEMONSTRATION.

Let $a:b::c:d$ represent any proportion. We are to prove that $ad=bc$. This proportion expressed as an equation is

$$\frac{a}{b}=\frac{c}{d} \qquad (1).$$

$$ad=bc \qquad (2)=(1)\times bd. \quad Q.E.D.$$

Or,

Put $a=rb$, then by the nature of a proportion $c=rd$. The proportion will then stand $rb:b::rd:d$.

Multiplying the extremes together, we have rbd.

Multiplying the means together, we have rbd.

These results are identical, therefore, the proposition is true.

REMARK.—This proposition furnishes the test of a proportion.

QUESTION.

Are 2, 4, 3, 7, in proportion?

SOLUTION.

Multiplying 2 by 7, we get 14, and 4 by 3, we get 12, which are not equal, therefore, by the foregoing proposition they are not in proportion.

QUESTIONS.

1. Are 3, 7, 8, 11 in proportion?

2. Are 8, 16, 4, 2 in proportion?

3. Are $2x$, $3x$, $4x$, $6x$ in proportion?

4. Are $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{16}$, $\frac{1}{8}$ in proportion?

5. Are $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{8}$, $\frac{1}{16}$ in proportion?

REMARK.—If any term of a proportion is unknown, put it equal to x, and form an equation by placing the product of the extremes equal to the product of the means, and then solve the equation to obtain the value of x. When one of the means is unknown, it is most convenient to put the product of the means equal to the product of the extremes.

PROPOSITION

(383.) 2. *When three numbers are in proportion, the product of the extremes is equal to the square of the mean.*

QUESTIONS.

1. Are 3, 4, 5 in proportion?

2. Are 3, 6, 12 in proportion?

3. Are $x, \sqrt{xy}, y$ in proportion?

4. Are ax, abx, bx in proportion?

5. Are ax, $x\sqrt{ab}$, bx in proportion?

PROPOSITION

(384.) 3. *When the product of two quantities is equal to the product of two other quantities, the four quantities may be expressed in the form of three different proportions.*

DEMONSTRATION.

Let $ad=bc$. We are then to prove that all of the following proportions are true,

$$a:b::c:d,$$
$$a:c::b:d,$$
$$b:a::d:c,$$

The equation $ad=bc$ may be put in the following forms:

$$\frac{a}{b}=\frac{c}{d},$$
$$\frac{a}{c}=\frac{b}{d},$$
$$\frac{b}{a}=\frac{d}{c},$$

and these three equations respectively give

$$a:b::c:d,$$
$$a:c::b:d,$$
$$b:a::d:c, \quad Q.\ E.\ D.$$

COROLLARY.—Since, on the supposition that $ad=bc$, we get the proportions

$$a:c::b:d$$
$$b:a::d:c,$$

we infer that these proportions are also true on the supposition that

$$a:b::c:d,$$

because this proportion gives

$$ad=bc.$$

From this fact we obtain the two following propositions:

PROPOSITION

(385.) 4. *When four quantities are in proportion, the first is to the third as the second is to the fourth.*

$$\text{If } a : b :: c : d,$$
$$\text{then } a : c :: b : d.$$

This is called proportion by *Alternation.*

PROPOSITION

(386.) 5. *When four quantities are in proportion, the second is to the first as the fourth is to the third.*

$$\text{If } a : b :: c : d,$$
$$\text{then } b : a :: d : c.$$

This is called proportion by *Inversion.*

PROPOSITION

(387.) 6. *When a couplet is common to two proportions, the other two couplets will constitute a proportion.*

$$\text{If } a : b :: m : n,$$
$$\text{and } c : d :: m : n,$$
$$\text{then } a : b :: c : d.$$

Let the student prove this.

PROPOSITION

(388.) 7. If $a : b :: c : d$, then are the following proportions true:

$$ma : mb :: mc : md$$
$$ma : mb :: c : d.$$
$$a : b :: mc : md$$
$$ma : b :: mc : d$$
$$a : mb :: c : md$$
$$ma : mb :: nc : nd$$
$$ma : nb :: mc : nd$$
$$\frac{a}{m} : \frac{b}{m} :: \frac{c}{m} : \frac{d}{m}$$
$$\frac{a}{m} : \frac{b}{m} :: c : d$$
$$a : b :: \frac{c}{m} : \frac{d}{m}$$
$$\frac{a}{m} : b :: \frac{c}{m} : d$$
$$a : \frac{b}{m} :: c : \frac{d}{m}$$

Let the student prove these proportions to be true by an application of Prop. 1, **(382.)**

PROPOSITION

(389.) 8. *When four quantities are in proportion, the sum of the first and second is to the second as the sum of the third and fourth is to the fourth.*

DEMONSTRATION.

Let $a : b :: c : d$ (1). We are to prove that
$a+b : b :: c+d : d$
$ad = bc$ (2)=(1) by Prop. 1, **(382.)**
$ad+bd = bc+bd$ (3)=(2) with bd added to both members.
$(a+b)d = b(c+d)$ (4)=(3) factored.

By Prop. 3, **(384)** $a+b : b :: c+d : d$. *Q. E. D.*

This and the derivative proportion in the following proposition is called proportion by *Composition.*

PROPOSITION

(390.) 9. *When four quantities are in proportion, the sum of the first and second is to the first as the sum of the third and fourth is to the third.*

Let the student prove this.

PROPOSITION

(391.) 10. *When four quantities are in proportion, the difference between the first and second is to the second as the difference between the third and fourth is to the fourth.*

DEMONSTRATION.

Let $a : b :: c : d$ (1). We are to prove that
$a-b : b :: c-d : d$
$ad = bc$ (2)=(1) by Prop. 1, **(382.)**
$ad-bd = bd-bc$ (3)=(2) with bd subtracted from both members.
$(a-b)d = b(c-d)$ (4)=(3) factored.

By Prop. 3, **(384)** $a-b : b :: c-d : d$. *Q. E. D.*

This and the derivative proportion in the following proposition is called proportion by *Division.*

PROPOSITION

(392.) 11. *When four quantities are in proportion, the difference between the first and second is to the first as the difference between the third and fourth is to the third.*

Let the student prove this.

PROPOSITION

(393.) 12. *When four quantities are in proportion, the sum of the first and second is to their difference as the sum of the third and fourth is to their difference.*

DEMONSTRATION.

Let $a:b::c:d$. We are to prove that

$$a+b:a-b::c+d:c-d$$

By Prop. 8, **(389)** and Alternation, $a+b:c+d::b:d$

By Prop. 9, **(390)** " " $a-b:c-d::b:d$

then by Prop. 6, **(387)** " " $a+b:a-b::c+d:c-d$

Q. E. D.

PROPOSITION

(394.) 13. *In a continued proportion, any antecedent is to its consequent as the sum of all the antecedents is to the sum of all the consequents.*

DEMONSTRATION.

Let $a:b::c:d::e:f::g:h::$ &c. We are to prove that

$a:b::a+c+e+g$, &c. $: b+d+f+h$, &c.

By Prop. 1, **(382)** $\left\{\begin{array}{l} ad=bc \\ af=be \\ ah=bg \\ \text{\&c.}=\text{\&c.} \end{array}\right.$

Whence, $ad+af+ah$, &c.$=bc+be+bg$, &c.

Adding ab, we have $ab+ad+af+ah$, &c.$=ab+bc+be+bg$, &c.

$a(b+d+f+h$, &c.$)=b(a+c+e+g$, &c.$)$

By Prop. 3. **(384)** $a:b::a+c+e+g$, &c. $: b+d+f+h$, &c.

Q. E. D.

PROPOSITION

(395.) 14. *If the corresponding terms in two proportions be multiplied together the products will constitute a proportion.*

DEMONSTRATION.

Let $a:b::c:d$,

and $m:n::p:q$.

We are to prove that $am:bn::cp:dq$.

By Prop. 1, **(382.)** $\begin{cases} ad=bc, \\ mq=np, \end{cases}$

Multiplying $am \cdot dq = bn \cdot cp$.

By Prop. 3, **(384.)** $am:bn::cp:dq$. *Q. E. D.*

PROPOSITION

(396.) 15. *If the terms of one proportion be divided by the corresponding terms of another proportion, the quotients will constitute a proportion.*

Let the student prove this.

PROPOSITION

(397.) 16. *If four quantities be in proportion, their like powers or roots will be in proportion.*

DEMONSTRATION.

Let $a:b::c:d$.

We are to prove that $\begin{cases} a^m:b^m::c^m:d^m, \\ a^{\frac{1}{m}}:b^{\frac{1}{m}}::c^{\frac{1}{m}}:d^{\frac{1}{m}}. \end{cases}$

By Prop. 1, **(382.)** $ad=bc$,

By involution $a^m d^m = b^m c^m$,

By evolution $a^{\frac{1}{m}}d^{\frac{1}{m}} = b^{\frac{1}{m}}c^{\frac{1}{m}}$.

Whence by Prop. 3, **(384.)** $\left\{\begin{array}{l} a^m:b^m::c^m:d^m, \\ a^{\frac{1}{m}}:b^{\frac{1}{m}}::c^{\frac{1}{m}}:d^{\frac{1}{m}}. \end{array}\right\}$ *Q. E. D.*

PROPOSITION

(398.) 17. *If three quantities are in proportion, the first is to the third in the duplicate ratio of the first to the second, that is as the square of the first is to the square of the second.*

If $a:b::b:c$, prove that $a:c::a^2:b^2$.

PROPOSITION

(399.) 18. *If four quantities are in continued proportion, the first is to the fourth in the triplicate ratio of the first to the second.*

If $a:b::b:c::c:d$, prove that $a:d::a^3:b^3$.

PROPOSITION

(400.) 19. If $m:n::p:q$ and $am:bn::c:d$, then $ap:bq::c:d$.

If $m:n::p:q$ and $a:b::mc:nd$, then $a:b::pc:qd$.

If $m:n::p:q$ and $am:b::\quad:d$, then $ap:b::cq:d$.

If $m:n::p:q$ and $a:bm::c:dn$, then $a:bp::c:dq$.

PROPOSITION

(401.) 20. If $\left\{\begin{matrix} a:b::c:d \\ a:e::c:f \end{matrix}\right\}$ then $a:b\pm e::c:d\pm f$.

If $\left\{\begin{matrix} a:b::c:d \\ e:b::f:d \end{matrix}\right\}$ then $a\pm e:b::c\pm f:d$.

PROPOSITION

(402.) 21. *If the two consequents of four quantities in proportion be increased or diminished by quantities which have the same ratio as the antecedents, the resulting quantities and the antecedents will be in proportion.*

If $\left\{\begin{matrix} a:b::c:d \\ a:c::m:n \end{matrix}\right\}$ then $a:c::b\pm m:d\pm n$.

HARMONICAL PROPORTION.

(403.) Three quantities are in *harmonical proportion*, when the first is to the third, as the difference between the first and the second is to the difference between the second and third.

The quantities a, b, and c are in *harmonical proportion* when

$$a:c:a\sim b:b\sim c.$$

(404.) Four quantities are in *harmonical proportion*, when the first is to the fourth, as the difference between the first and second is to the difference between the third and fourth.

The quantities a, b, c, and d are in harmonical proportion when

$$a:d::a\sim b:c\sim d.$$

PROBLEM.

(405.) To find an *harmonical mean* between a and c.

SOLUTION.

Let $x=$ the required *harmonical mean*. Since a, x, and c are in harmonical proportion, we have $a:c::a-x:x-c$.

$$ax-ac=ac-cx,$$
$$(a+c)x=2ac,$$
$$x=\frac{2ac}{a+c}, \text{ the mean to be found.}$$

PROBLEM

1. Given the first and second of three quantities in *harmonical proportion* to find the third.

PROBLEM

2. Given the second and third of three quantities in harmonical proportion to find the first.

PROBLEM

3. Given the first three of four quantities in harmonical proportion to find the fourth.

PROBLEM

4. Given the last three of four quantities in harmonical proportion to find the first.

PROBLEM

5. Given the first and last and one of the middle quantities of four quantities in harmonical proportion to find the other middle quantity.

HARMONICAL PROGRESSION.

(406.) AN HARMONICAL PROGRESSION is a series of quantities, any consecutive three of which are in *harmonical proportion.*

PROPOSITION

(407.) 1. *The reciprocals of a series of quantities in harmonical progression are in arithmetical progression.*

DEMONSTRATION.

Let a, b, c, d, e, f, &c., be an *harmonical* progression. We are to prove that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}, \frac{1}{e}, \frac{1}{f}$, &c. is an *arithmetical* progression.

If we prove that $\begin{cases} \frac{1}{a}+\frac{1}{c}=\frac{2}{b}, \\ \frac{1}{b}+\frac{1}{d}=\frac{2}{c}, \\ \frac{1}{c}+\frac{1}{e}=\frac{2}{d}, \\ \frac{1}{d}+\frac{1}{f}=\frac{2}{e}, \\ \text{\&c., \&c.,} \end{cases}$

we shall establish the truth of the proposition.

We have $b=\frac{2ac}{a+c}$, or $\frac{b}{2}=\frac{ac}{a+c}$, or $\frac{2}{b}=\frac{a+c}{ac}=\frac{1}{c}+\frac{1}{a}$.

We have $b=\frac{2ac}{a+c}$, which gives $\frac{1}{a}+\frac{1}{c}=\frac{2}{b}$,

" " $c=\frac{2bd}{b+d}$, " " $\frac{1}{b}+\frac{1}{d}=\frac{2}{c}$,

" " $d=\frac{2ce}{c+e}$, " " $\frac{1}{c}+\frac{1}{e}=\frac{2}{d}$,

" " $e=\frac{2df}{d+f}$, " " $\frac{1}{d}+\frac{1}{f}=\frac{2}{e}$. *Q. E. D.*

ANOTHER DEMONSTRATION.

We are to prove that $\begin{cases} \frac{1}{a}-\frac{1}{b}=\frac{1}{b}-\frac{1}{c}, \\ \frac{1}{b}-\frac{1}{c}=\frac{1}{c}-\frac{1}{d}, \\ \frac{1}{c}-\frac{1}{d}=\frac{1}{d}-\frac{1}{e}, \\ \frac{1}{d}-\frac{1}{e}=\frac{1}{e}-\frac{1}{f}. \\ \text{\&c., \quad \&c.} \end{cases}$

By the nature of the progression, we have $\begin{cases} a:c::a-b:b-c, \\ b:d::b-c:c-d, \\ c:e::c-d:d-e, \\ d:f::d-e:e-f. \end{cases}$

Whence, we get $\begin{cases} ab-ac=ac-bc, \\ bc-bd=bd-cd, \\ cd-ce=ce-de, \\ de-df=df-ef, \end{cases}$

Dividing respectively by $\left\{\begin{array}{l} abc, \text{ gives } \frac{1}{c}-\frac{1}{b}=\frac{1}{b}-\frac{1}{a}, \\ bcd, \text{ gives } \frac{1}{d}-\frac{1}{c}=\frac{1}{c}-\frac{1}{b}, \\ cde, \text{ gives } \frac{1}{e}-\frac{1}{d}=\frac{1}{d}-\frac{1}{c}, \\ def, \text{ gives } \frac{1}{f}-\frac{1}{e}=\frac{1}{e}-\frac{1}{d}. \end{array}\right.$

These equations by transposition,

become $\left\{\begin{array}{l} \frac{1}{a}-\frac{1}{b}=\frac{1}{b}-\frac{1}{c}, \\ \frac{1}{b}-\frac{1}{e}=\frac{1}{c}-\frac{1}{d}, \\ \frac{1}{c}-\frac{1}{d}=\frac{1}{d}-\frac{1}{e}, \\ \frac{1}{d}-\frac{1}{e}=\frac{1}{e}-\frac{1}{f}. \end{array}\right.$ *Q. E. D.*

The converse of the preceding proposition is evidently true, therefore, we have

PROPOSITION

(408.) 2. *The reciprocals of a series of quantities in arithmetical progression, will constitute an harmonical progression.*

Since 1, 2, 3, 4, 5, 6, &c., is an arithmetical series, their reciprocals

$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$, &c., is an harmonical series.

The fractions in this series are in the ratio of

60, 30, 20, 15, 12, 10, &c.,

which must also, be an *harmonic series.*

REMARK.—It is a principle in music that the longer a string is, the lower is the sound produced by its vibration. If then, we have musical strings of equal weight and tension which are in the ratio of 60, 30, 20, 15, 12, 10, and call the sound produced by the vibrations of the string whose length is 60, the key note, the sounds produced by the string whose length is 30, will be the octave of this key-note; by the string 20 the fifth of this octave, or the twelfth of the key-note: by the string 15, the octave of the octave, or the double octave of the key-note; by the string 12 the third of this double octave or the seventeenth of the key-note; by the string 10 the fifth of this double octave or the nineteenth of the key-note. The simultaneous vibration of these will produce what is called *harmony;* hence, the name *harmonical progression.*

PROBLEM.

(409.) To find any number of harmonic means between two quantities.

SOLUTION.

Let it be required to find m harmonic means between a and c.

By Prop. 1, **(407.)**, we have $\frac{1}{b}\ldots\ldots\ldots\frac{1}{c}$, an arithmetical progression of $m+2$ terms.

By substituting in the formula $l=a\pm(n-1)d$, we have

$$\frac{1}{c}=\frac{1}{b}\pm(m+1)d,$$

$$\frac{b-c}{bc}=\pm(m+1)d,$$

$$d=\frac{b-c}{\pm(m+1)bc}.$$

Having found the common difference, we are now able to insert m arithmetical means in the arithmetical series.

$$\frac{1}{b}\cdots\cdots\frac{1}{c}.$$

The series becomes

$$\div\frac{1}{b}\cdot\frac{1}{b}+\frac{b-c}{\pm(m+1)bc}\cdot\frac{1}{b}+\frac{2(b-c)}{\pm(m+1)bc}\cdot\frac{1}{b}+\frac{3(b-c)}{\pm(m+1)bc}\cdot$$
$$\frac{1}{b}+\frac{4(b-c)}{\pm(m+1)bc}\cdots\frac{1}{c},$$

or $$\div\frac{1}{b}\cdot\frac{mc+b}{(m+1)bc}\cdot\frac{mc+2b-c}{(m+1)bc}\cdot\frac{mc+3b-2c}{(m+1)bc}\cdot\frac{mc+4b-3c}{(m+1)bc}\cdots\frac{1}{c}.$$

Hence, the harmonic series is

$$b,\ \frac{(m+1)bc}{mc+b},\ \frac{(m+1)bc}{(m-1)c+2b},\ \frac{(m+1)bc}{(m-2)c+3b},\ \frac{(m+1)bc}{(m-3)c+4b}\cdots\frac{1}{c}.$$

EXAMPLES.

1. Find an harmonic mean between 3 and 6 *Ans.* 4.

2. Find an harmonic mean between $x+y$ and $x-y$. *Ans.* $\frac{x^2-y^2}{x}$.

3. Find an harmonic mean between $\frac{1}{x+y}$ and $\frac{1}{x-y}$. *Ans.* $\frac{1}{x}$.

4. Find the third of three quantities in harmonical proportion, the first and second being 3 and 4. *Ans.* 6.

5. Find the first of three quantities in harmonical proportion, the second and third being 144 and 104. *Ans.* 234.

6. Find the fourth of four quantities in harmonic proportion, the first three being 2, 3, and 8. *Ans.* 16.

7. Find the third of four quantities in harmonical proportion, the first being 10, the second 12, and the fourth 15. *Ans.* 12.

8. Find the second of four quantities in harmonical proportion, the first being 5, the third 9, and the fourth 15. *Ans.* 7.

9. Find the first of four quantities in harmonical proportion, the second being 6, the third 9, and the fourth 15. *Ans.* $4\frac{2}{7}$.

10. Find a harmonic mean between 50 and 100. *Ans.* $66\frac{2}{3}$,

11. Find a harmonic mean between 25 and 50. *Ans.* $33\frac{1}{3}$.

12. Find a harmonic mean between $12\frac{1}{2}$ and 25. *Ans.* $16\frac{2}{3}$.

13. Find two harmonic means between $1\frac{1}{5}$ and 3. *Ans.* $1\frac{1}{2}$ and 2.

14. Find three harmonic means between 10 and 30.

Ans. 12, 15, and 20.

15. Find three harmonic means between 315 and 35.

Ans. 105, 63, and 45.

16. Find fourteen harmonic means between $\frac{1}{4}$ and 4.

Ans. $\frac{4}{15}$, $\frac{2}{7}$, $\frac{4}{13}$, $\frac{1}{3}$, $\frac{4}{11}$, $\frac{2}{5}$, $\frac{4}{9}$, $\frac{1}{2}$, $\frac{4}{7}$, $\frac{2}{3}$, $\frac{4}{5}$, 1, $1\frac{1}{3}$, and 2.

17. Find the fifth term of an harmonical progression whose first term is 60 and second term 21. *Ans.* $7\frac{7}{59}$.

18. Find the unknown terms of an harmonical progression consisting of 12 terms, the first being 4 and the fourth 1.

Ans. 2, $1\frac{1}{3}$, $\frac{4}{5}$, $\frac{2}{3}$, $\frac{4}{7}$, $\frac{1}{2}$, $\frac{4}{9}$, $\frac{2}{5}$, $\frac{4}{11}$, and $\frac{1}{3}$.

19. Find the resulting proportion when a, b, c are in arithmetical progression, and b, c, d in harmonic proportion. *Ans.* $a : b :: c : d$.

20. Find the nth term of an harmonical progression, a and b being the first two terms.

$$Ans.\ \frac{ab}{a(n-1)-b(n-2)}.$$

PROBLEMS IN PROPORTION.

PROBLEM.

(410.) There are two numbers whose product is 24, and the difference of their cubes is to the cube of their difference as 19 is to 1. What are the numbers?

SOLUTION.

Let $x =$ the greater number,
and $y =$ the lesser number.

By 1st condition, $x^3-y^3 : (x-y)^3 :: 19 : 1$

$x^2+xy+y^2 : x^2-2xy+y^2 :: 19 : 1$	Prop. 7. (9)
$3xy : x^2+xy+y^2 :: 18 : 19$	Prop. 11.
$xy : x^2+xy+y^2 :: 6 : 19$	Prop. 7. (11)
$xy : x^2+2xy+y^2 :: 6 : 25$	Prop. 9 and Prop. 5.
$4xy : x^2+2xy+y^2 :: 24 : 25$	Prop. 7. (4)
$(x+y)^2 : (x-y)^2 :: 25 : 1$	Prop. 10 and Prop. 5.
$x+y : x-y :: 5 : 1$	Prop. 16.
$2x : 2y :: 6 : 4$	Prop. 12.
$x : y :: 3 : 2$	Prop. 7. (8)
$3y=2x$	Prop. 1.
$y=\frac{2}{3}x$	

By 2d condition, $xy=24$

$$\frac{2}{3}x^2=24$$
$$x^2=36$$
$$x=\pm 6$$
$$y=\pm 4$$

QUESTIONS.

1. There are two numbers whose product is 135, and the difference of their squares is to the square of their difference as 4 to 1. What are the numbers? *Ans.* 15 and 9.

2. There are two numbers which are to each other in the duplicate ratio of 4 to 3, and 24 is a mean proportional between them. What are the numbers? *Ans.* 32 and 18.

3. There are two numbers whose sum is 24, and their product is to the sum of their squares as 3 to 10. What are the numbers? *Ans.* 18 and 6.

4. There are three numbers which are to each other as 3 to 2. If 6 be added to the greater and subtracted from the lesser, the sum will be to the remainder as 3 to 1. What are the numbers?

Ans. 24 and 16.

5. There are two numbers whose sum is 60, and their product is to the sum of their squares as 2 to 5. What are the numbers?

Ans. 40 and 20.

6. The number 20 is divided into two parts, which are to each other in the duplicate ratio of 3 to 1. What is the mean proportional between these parts? *Ans.* 6.

7. If $\frac{a^2-x^2}{b}=4a$, show that $a+x:2a::2b:a-x$.

8. If $(a+x)^2:(a-x)^2::x+y:x-y$, show that $a:x::\sqrt{2a-y}:\sqrt{y}$.

9. If $x:y::a^3:b^3$ and $a:b::\sqrt[3]{c+x}:\sqrt[3]{d+y}$ show that $dx=cy$.

10. If $x^2:y^2::36:25$ and $2x+y:x+2$ in a ratio compounded of the ratios of $17:2$ and $2:7$, what are the values of x and y?

Ans. $x=12$, and $y=10$.

11. A person has British wine at 5s. per gallon, with which he wishes to mix spirits at 11s. per gallon, in such proportion that by selling the mixture at 9s. a gallon he may gain 35 per cent. What is the necessary proportion?

Ans. 13 gallons of wine to 5 of spirits.

12. What number is that to which if 3, 8, and 17 be severally added, the first sum shall be to the second as the second to the third?

Ans. $3\frac{1}{4}$.

13. A merchant having mixed a certain number of gallons of brandy and water, found that if he had mixed 6 gallons more of each there would have been 7 gallons of brandy to every 6 gallons of water, but if he had mixed 6 gallons less of each there would have been 6 gallons of brandy to every 5 gallons of water. How much of each did he mix?

Ans. 78 gallons of brandy and 66 of water.

14. *A* and *B* speculate in trade with different sums. *A* gains \$150, *B* loses \$50, and now *A*'s stock is to *B*'s as 3 to 2; but, had *A* lost \$50 and *B* gained \$100, then *A*'s stock would have been to *B*'s as 5 to 9. What was the stock of each?

Ans. *A*'s \$300 and *B*'s \$350.

15. What are the two parts of 14 of which the greater divided by the less is to the less divided by the greater as 16 to 9.

Ans. 8 and 6.

16. In a mixture of rum and brandy, the difference between the quantities of each is to the quantity of brandy as 100 is to the number of gallons of rum; and the same difference is to the quantity of rum as 4 is to the number of gallons of brandy. How many gallons are there of each? *Ans.* 25 gallons of rum and 5 of brandy.

17. There is a number consisting of three digits, the first of which is to the second as the second is to the third; the number itself is to the sum of its digits as 124 to 7; and if 594 be added to it the digits will be inverted. What is the number? *Ans.* 248.

18. A corn-factor mixes wheat which cost 10s. a bushel with barley which cost 4s. a bushel, in such proportion as to gain $43\frac{3}{4}$ per cent. by selling the mixiure at 11s. a bushel. What is the proportion? *Ans.* 14 bushels of wheat to 9 of barley.

19. What two numbers are those whose sum, difference, and product, are as the numbers 3, 2, and 5, respectively. *Ans.* 10 and 2.

20. What two numbers are those whose sum, difference, and product, are as the numbers s, d, and p respectively?

Ans. $\dfrac{2p}{s+d}$ and $\dfrac{2p}{s-d}$.

21. There are two numbers in the proportion of $\frac{1}{2}$ to $\frac{2}{3}$, and such, that if they be increased respectively by 6 and 5, they will be to each other as $\frac{2}{5}$ to $\frac{1}{2}$. What are the numbers? *Ans.* 30 and 40.

22. There is a number, the sum of whose digits is to the number itself as 4 to 13; and if the digits be inverted, their difference will be to the number expressed as 2 to 31. What is the number?

Ans. 39.

23. The difference of the cubes of two numbers is to the cube of their difference as 61 is to 1, and the product of the numbers is 320. What are the numbers? *Ans.* ± 20 and ± 16.

24. The sum of the cubes of two numbers is to the difference of their cubes as 559 to 127, and the square of the first multiplied by the second is 294. What are the numbers? *Ans.* 7 and 6.

25. The difference of the cubes of two numbers is to their product multiplied by their difference as 7 is to 2, and the sum of the numbers is 6. What are the numbers? *Ans.* 4 and 2.

26. The difference of two numbers multiplied by the first is to the difference multiplied by the second as 3 to 7, and the first multiplied by the square of the second is 147. What are the numbers?

Ans. 3 and 7.

27. The sum of the squares of two numbers is to the difference of their squares as 17 is to 8, and the first number multiplied by the square of the second is 45. What are the numbers?

Ans. 5 and 3.

28. The sum of two numbers is to the difference of their squares as 1 is to 4, and the product of the numbers is 21. What are the numbers? *Ans.* ± 7 and ± 3.

29. The sum of three numbers in geometrical progression is 52, and the sum of the extremes is to the mean as 10 to 3. What are the numbers? *Ans.* 4, 12, and 36.

30. The difference of two numbers is to 6, as the greater is to the less, and as their sum is to 42. What are the numbers?

Ans. 32 and 24.

31. Given $\dfrac{a-\sqrt{a^2-x^2}}{a+\sqrt{a^2-x^2}}=b$ to find x by changing the equation into a proportion. *Ans.* $x=\dfrac{\pm 2a\sqrt{b}}{b+1}$.

32. Given $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=b$ to find x by changing the equation into a proportion. *Ans.* $x=\dfrac{2ab}{b^2+1}$.

33. Given $\dfrac{\sqrt[3]{x+1}+\sqrt[3]{x-1}}{\sqrt[3]{x+1}-\sqrt[3]{x-1}}=2$ to find x by changing the equation into a proportion. *Ans.* $x=\frac{14}{13}$.

34. Given $\dfrac{\sqrt{4x+1}+\sqrt{4x}}{\sqrt{4x+1}-\sqrt{4x}}=9$ to find x by changing the equation into a proportion. *Ans.* $x=\frac{4}{9}$.

35. Given $\dfrac{a+x+\sqrt{2ax+x^2}}{a+x}=b$ to find x by changing the equation into a proportion. *Ans.* $x=\dfrac{a(\pm 1-\sqrt{2b-b^2})}{\sqrt{2b-b^2}}$.

CHAPTER XVIII.

SERIES.

(411.) There are many kind of series. Arithmetical and geometrical series have already been discussed. A series may result from dividing the numerator of a fraction by its denominator, or from involution or evolution.

THE EXPANSION OF SERIES.

(412.) Quantities or algebraic expressions may be expanded or developed in four ways; namely,

The Method of Division,
" " " *Undetermined Coefficients*,
" " " *Involution*,
and " " " *Evolution*.

THE METHOD OF DIVISION.

PROBLEM

1. Expand $\frac{1}{1+a}$ into a series.

SOLUTION.

By dividing 1 by $1+a$, we obtain $1-a+a^2-a^3+$, &c.

$$\begin{array}{l} 1+a)1 \quad (1-a+a^2-a^3+, \text{ \&c.} \\ \underline{1+a} \\ -a \\ \underline{-a-a^2} \\ \qquad a^2 \\ \qquad \underline{a^2+a^3} \\ \qquad\quad -a^3 \\ \qquad\quad \underline{-a^3-a^4} \\ \qquad\qquad a^4. \end{array}$$

Therefore, $\frac{1}{1+a}=1-a+a^2-a^3+a^4-a^5+a^6$, &c.

PROBLEM

2. Expand $\frac{1}{8}=\frac{1}{1+7}$ into a series.

SOLUTION.

We see that this fraction is the same as $\frac{1}{1+a}$ if we suppose a to be 7; therefore,

$$\frac{1}{8}=\frac{1}{1+7}=1-7+7^2-7^3, \text{ \&c.}$$

$$\frac{1}{8}=1-7+49-343, \text{ \&c.}$$

The student may not, at first sight, see how this series can be equal to $\frac{1}{8}$. But if the remainder is taken into consideration, the result may be verified. By considering the remainder, we have

$$\frac{1}{8}=1-7+49-343+\frac{2401}{8}.$$

$$\frac{1}{8}=-300+\frac{2401}{8},$$

$$\frac{1}{8}=\frac{-2400+2401}{8},$$

$$\frac{1}{8}=\frac{1}{8}.$$

This fraction may also be expanded into a finite series.

Thus, $\frac{1}{8}=\frac{1.000}{8}=\cdot125.\ \frac{1}{8}=\frac{1000}{8000}=\frac{125}{1000}=\frac{125}{1000}=\cdot125.$

This method furnishes an explanation of the arithmetical process of converting $\frac{1}{8}$ into a decimal fraction.

EXAMPLES.

1. Expand $\frac{1}{1-a}$ into a series.

Ans. $1+a+a^2+a^3+a^4+a^5+a^6+\ldots\ldots$

2. Expand $\frac{1-a}{1+a}$ into a series.

Ans. $1-2a+2a^2-2a^3+2a^4-2a^5+\ldots\ldots$

3. Expand $\frac{1}{0}=\infty$ into a series.

Ans. $1+1+1+1+1+1+1+1+1+\ldots\ldots\ldots$

4. Expand $\frac{1}{1-2}$ into a series.

Ans. $1+2+4+8+16+32+64+\ldots\ldots\ldots$

5. Expand $\frac{a}{a-b}$ into a series.

Ans. $1+\frac{b}{a}+\frac{b^2}{a^2}+\frac{b^3}{a^3}+\frac{b^4}{a^4}+\frac{b^5}{a^5}+\ldots\ldots\ldots$

6. Expand $\frac{a}{1-r}$ into a series.

Ans. $a+ar+ar^2+ar^3+ar^4+ar^5+\ldots\ldots\ldots$

7. Expand $\frac{a}{a+b}$ into a series.

Ans. $1-\frac{b}{a}+\frac{b^2}{a^2}-\frac{b^3}{a^3}+\frac{b^4}{a^4}-\frac{b^5}{a^5}+\ldots\ldots\ldots$

8. Expand $\frac{1+x}{1-x}$ into a series.

Ans. $1+2x+2x^2+2x^3+2x^4+\ldots\ldots\ldots$

9. Expand $\frac{1}{1-x+x^2}$ into a series.

Ans. $1+x-x^3-x^4+x^6+x^7-x^9-x^{10}+\ldots\ldots\ldots$

10. Expand 1 into a series.

$$\textit{Ans.}\quad \begin{cases} x+x+x+x+x+x+x+x+ \quad \ldots\ldots \\ x-x^2+x^3-x^4+x^5-x^6+x^7-x^8+\ldots\ldots \\ 1-\frac{1}{x}+\frac{1}{x^2}-\frac{1}{x^3}+\frac{1}{x^4}-\frac{1}{x^5}+\frac{1}{x^6}- \quad \ldots\ldots \end{cases}$$

NOTE.—These results may all be verified if the remainders are considered.

THE METHOD OF UNDETERMINED COEFFICIENTS.

(413.) The method of undetermined coefficients is a method by which algebraic fractions may be expanded or developed into a series, the terms of the series being arranged according to the ascending powers of one of the quantities considered as a variable. This method is based upon the following theorems.

THEOREM I.

(414.) 1. *If the series* $Ax^a+Bx^b+Cx^c+Dx^d+$, &c.$=A'x^{a'}+B'x^{b'}+C'x^{c'}+D'x^{d'}+$, &c., *for all possible values of* x, *the exponents and coefficients being finite quantities, and the exponents of* x *in each member being arranged in the order of their magnitudes, commencing with the least, then* $a=a'$; $b=b'$; $c=c'$; $d=d'$; &c.. and $A=A'$; $B=B'$; $C=C'$; $D=D'$; &c.

DEMONSTRATION.

Dividing both members by x^a, the equation becomes $A+Bx^{b-a}+Cx^{c-a}+Dx^{d-a}+$, &c.$=A'x^{a'-a}+B'x^{b'-a}+C'x^{c'-a}+D'x^{d'-a}+$, &c.

Since this equation is true for any value of x, we have a right to suppose x equal to zero. Making this supposition the first member reduces to A. If we suppose all the exponents in the second member to be finite, it would reduce to zero when x equals zero, and we should have $A=0$, which is impossible, as A is supposed to some finite be quantity. Therefore, it would be improper to consider all the exponents in the second member as finite, and hence, one or more of them must be zero. Since, a', b', c', d', &c., are all different, it is evident that a can not equal more than one of these quantities, and, therefore, of the exponents $a'-a$; $b'-a$; $c'-a$; $d'-a$; &c., but one can equal zero.

It now remains for us to decide which exponent it is that equals zero. Suppose it to be $b'-a$, or the exponent of x in the second term, and then for x equal zero all the terms after the second must vanish, and there would result the equation

$$A=A'x^{a'-a}+Bx^0.$$

But since $x^0=1$, $\qquad A=A'x^{a'-a}+B.$

Since $a=b'$ and a' is less than b', $a'-a=$ a negative quantity which put equal $-n$, and the equation becomes

$$A=A'x^{-n}+B,$$

$$A=\frac{A'}{x^n}+B,$$

Supposing $x=0$ $\qquad A=\frac{A'}{0}+B,$

$$A-B=\infty.$$

But $A-B$ must be equal to a positive or negative quantity, or zero, therefore $A-B=\infty$ is impossible, which shows that it is improper to make $b'-a=0$.

As the same reasoning applies to all the exponents of x in the

second member but the first, or $a'-a$, it follows that $a'-a$ must equal zero, and the equation becomes

$$A=A'x^0,$$
$$A=A'.$$

Suppressing the equal terms $A=A'x^{a'-a}$ the equation becomes

$$Bx^{b-a}+Cx^{c-a}+Dx^{d-a}+, \text{\&c.},=B'x^{b'-a}+C'x^{c-a}+D'x^{d'-a}+, \text{\&c.}$$

Dividing by x^{b-a}, we have

$$B+Cx^{c-b}+Dx^{d-b}+, \text{\&c.},=B'x^{b'-b}+C'x^{c-b}+D'x^{d'-b}+, \text{\&c.}$$

Then for $x=$ zero, we have when we make $b'-b=$ zero, or $b'=b$

$$B=B'x^0,$$
$$B=B'.$$

Thus we may go on and prove that $c=c'$ and $C=C'$; $d=d'$ and $D=D'$, &c., which proves the theorem.

THEOREM II.

(415.) If $Ax^a+Bx^b+Cx^c+Dx^d+$, &c. $=0$ for all possible values of x, a being less than b, b less than c, c less than d, &c., each of the coefficients must be equal to zero.

DEMONSTRATION.

Dividing by x^a, we have

$$A+Bx^{b-a}+Cx^{c-a}+Dx^{d-a}+, \text{\&c.} =0.$$

If now we make $x=$ zero, we obtain

$$A=0.$$

Suppressing $A=0$, and dividing by x^{b-a}, we obtain

$$B+Cx^{c-b}+Dx^{d-b}+, \text{\&c.} =0,$$

in which if x is put equal to zero the equation reduces to

$$B=0$$

In the same way, we can get $C=0$, $D=0$, &c., which was what was to be proved.

Let us now apply the method of undetermined coefficients to the expansion of series.

PROBLEM

(416.) 1. Expand $\dfrac{1}{1-2x+x^2}$ into a series.

SOLUTION.

By an inspection of this fraction we see that by division the first term of the quotient would not contain x but the second term would.

Assume, then, $\frac{1}{1-2x+x^2}=A+Bx+Cx^2+Dx^3+Ex^4+\ldots\ldots$

Clearing of fractions, we have

$$\begin{aligned}1=A+\ &Bx+\ Cx^2+\ Dx^3+\ Ex^4+\ldots\ldots\\ &-2Ax-2Bx^2-2Cx^3-2Dx^4-\ldots\ldots\\ &\qquad\qquad+\ Ax^2+\ Bx^3+\ Cx^4+\ldots\ldots\end{aligned}$$

Transposing, the equation becomes

$$1+2Ax+2Bx^2+2Cx^3+2Dx^4+\ldots\ldots=A+Bx+(A+C)x^2+(B+D)x^3+(C+E)x^4+\ldots\ldots$$

According to Theorem 1 (**414**), the corresponding coefficients in these series must be equal.

$$\therefore \left\{\begin{array}{r}A=1\\ B=2A\\ A+C=2B\\ B+D=2C\\ C+E=2D\\ \text{\&c.}\end{array}\right\}, \text{ whence } \left\{\begin{array}{l}A=1\\ B=2\\ C=3\\ D=4\\ E=5\\ \text{\&c.}\end{array}\right.$$

Substituting these values in the assumed series, we have

$$\frac{1}{1-2x+x^2}=1+2x+3x^2+4x^3+5x^4+6x^5+7x^6+\ldots\ldots\ldots$$

This fraction might have been developed into a series with more facility by division.

The student should be very careful in assuming the series of undetermined coefficients. This may be illustrated by the following

PROBLEM

2. Expand $\frac{2}{3x^2-2x^3}$ into a series.

SOLUTION.

By division we see that the first term of the expansion must be

$$=\frac{2}{3x^2}=\frac{2}{3}x^{-2}, \text{ therefore, assume}$$

$$\frac{2}{3x^2-2x^3}=Ax^{-2}+Bx^{-1}+Cx^0+Dx+Ex^2+Fx^3+\text{\&c.}$$

Clearing of fractions and transposing, we have

$$2+2Ax+2Bx^2+2Cx^3+2Dx^4+2Ex^5+, \text{\&c.}=3A+3Bx+3Cx^2+3Dx^3+3Ex^4+3Fx^5+\text{\&c.}$$

By equating, like coefficients,

we have $\left\{\begin{array}{l} 3A=2 \\ 3B=2A \\ 3C=2B \\ 3D=2C \\ 3E=2D \\ 3F=2E \\ \&c. \end{array}\right\}$, whence $\left\{\begin{array}{l} A=\frac{2}{3} \\ B=\frac{4}{9} \\ C=\frac{8}{27} \\ D=\frac{16}{81} \\ E=\frac{32}{243} \\ F=\frac{64}{729} \\ \&c. \end{array}\right.$

By substituting these values in the assumed series, we get

$$\frac{2}{3x^2-2x^3}=\frac{2}{3x^2}+\frac{4}{9x}+\frac{8}{27}+\frac{16x}{81}+\frac{32x^2}{243}+\frac{64x^3}{729}+\ \&c.,\ \text{or}$$

$$\frac{2}{3x^2-2x^3}=\frac{2}{3x^2}+\frac{2^2}{3^2x^2}+\frac{2^3}{3^3}+\frac{2^4x}{3^4}+\frac{2^5x^2}{3^5}+\frac{2^6x^3}{3^6}+\ \&c.,$$ whence the law of the terms is apparent.

Since $\frac{2}{3x^2-2x^3}=\frac{2}{x^2}\left(\frac{1}{3-2x}\right)$, we might have developed $\frac{1}{3-x}$ by assuming it equal to $A+Bx+Cx^2+Dx^3+$ &c., and then multiplying the result by $\frac{2}{x^2}$ for the development of $\frac{2}{3x^2-2x^3}$.

PROBLEM

3. Expand $(1+x)^{\frac{1}{2}}$ into a series.

SOLUTION.

Assume $(1+x)^{\frac{1}{2}}=A+Bx+Cx^2+Dx^3+Ex^4+\ldots\ldots$

Squaring

$$\begin{aligned} 1+x+0x^2+0x^3+0x^4+\&c. &= A^2+2ABx+2ACx^2+2ADx^3+2AEx^4+\ldots\ldots \\ &\quad B^2x^2+2BCx^3+2BDx^4+\ldots\ldots \\ &\quad +C^2x^4+\ldots\ldots \end{aligned}$$

Equating like coefficients

we have $\left\{\begin{array}{r} A^2=1 \\ 2AB=1 \\ 2AC+B^2=0 \\ 2AD+2BC=0 \\ 2AE+2BD+C^2=0 \\ \&c. \end{array}\right\}$, whence $\left\{\begin{array}{l} A=1 \\ B=\frac{1}{2} \\ C=-\frac{1}{8} \\ D=\frac{1}{16} \\ E=-\frac{5}{128} \\ \&c. \end{array}\right.$

Substituting these values in the assumed series, we get

$$\sqrt{1+x}=1+\tfrac{1}{2}x-\tfrac{1}{8}x^2+\tfrac{1}{16}x^3-\tfrac{5}{128}x^4+\ldots\ldots$$

But we had a right to assume

$$\sqrt{1+x}=-A-Bx-Cx^2-Dx^3-Ex^4-\ldots\ldots$$

By squaring we should obtain the same equation as before, and hence the same values for A, B, C, D, E, &c., which being substituted in the assumed expression would give

$$\sqrt{1+x}=-1-\tfrac{1}{2}x+\tfrac{1}{8}x^2-\tfrac{1}{16}x^3+\tfrac{5}{128}x^4-\ldots\ldots$$

From these two results we see that

$$\sqrt{1+x}=\pm1\pm\tfrac{1}{2}x\mp\tfrac{1}{8}x^2\pm\tfrac{1}{16}x^3\mp\tfrac{5}{128}x^4\pm\ldots\ldots$$

the upper row of signs being taken for one result and the lower row for the other; or we may write it thus,

$$\sqrt{1+x}=\pm(1+\tfrac{1}{2}x-\tfrac{1}{8}x^2+\tfrac{1}{16}x^3-\tfrac{5}{128}x^4+\ldots\ldots)$$

DECOMPOSITION OF RATIONAL FRACTIONS.

(417.) The method of undetermined coefficients may be employed in the decomposition of rational fractions, and for this purpose it is often used in the Integral Calculus.

PROBLEM.

Decompose the fraction $\dfrac{5x-19}{x^2-8x+15}$.

SOLUTION.

Since $\dfrac{5x-19}{x^2-8x+15}=\dfrac{5x-19}{(x-3)(x-5)}$ let us assume that

$$\frac{5x-19}{(x-3)(x-5)}=\frac{A}{x-3}+\frac{B}{x-5}$$

$$\frac{5x-19}{(x-3)(x-5)}=\frac{A(x-5)+B(x-3)}{(x-3)(x-5)}$$

$$5x-19=A(x-5)+B(x-3)$$

$$(5A+3B)x^0+5x=19x^0+(A+B)x$$

Equating like coefficients of x,

$$\text{We have }\begin{cases}5A+3B=19,\\ A+\ B=5,\end{cases}$$

$$\begin{aligned}5A+3B&=19,\\ 3A+3B&=15.\\ \hline 2A&=4,\\ A&=2,\\ B&=3.\end{aligned}$$

$$\therefore\quad \frac{5x-19}{x^2-8x+15}=\frac{2}{x-3}+\frac{3}{x-5}.$$

NOTE.—The values of A and B might have been determined in a different manner. The equation

$$5x-19=A(x-5)+B(x-3).$$

Since this equation is true for all values of x, let us assume it to be 3, and we have $\quad 15-19=-2A$, or $A=2$.

If $x=5$ we have $25-19=\quad 2B$, or $B=3$.

EXAMPLES.

1. Expand $\dfrac{1-x}{1+x+x^2}$ into a series.

$$Ans.\ 1-2x+x^2+x^3-2x^4+x^5+x^6-2x^7+\ \dots\dots$$

2. Expand $\dfrac{a}{c+bx}$ into a series.

$$Ans.\ \frac{a}{c}-\frac{ab}{c^2}x+\frac{ab^2}{c^3}x^2-\frac{ab^3}{c^4}x^3+\frac{ab^4}{c^5}x^4-\ \dots\dots$$

3. Expand $\dfrac{1}{(a-x)^2}$ into a series.

$$Ans.\ \frac{1}{a^2}+\frac{2x}{a^3}+\frac{3x^2}{a^4}+\frac{4x^3}{a^5}+\frac{5x^4}{a^6}+\ \dots\dots$$

4. Expand $\dfrac{a-bx}{a+cx}$ into a series.

$$Ans.\ 1-(b+c)\frac{x}{a}+c(b+c)\frac{x^2}{a^2}-c^2(b+c)\frac{x^3}{a^3}+\ \dots\dots$$

5. Expand $\dfrac{1-x}{1-2x-3x^2}$ into a series.

$$Ans.\ 1+x+5x^2+13x^3+41x^4+121x^5+365x^6+\ \dots$$

6. Expand $\dfrac{d}{b-ax}$ into a series.

$$Ans.\ \frac{d}{b}+\frac{adx}{b^2}+\frac{a^2dx^2}{b^3}+\frac{a^3dx^3}{b^4}+\frac{a^4dx^4}{b^5}+\ \dots\dots$$

7. Expand $(a-x)^{-1}$ into a series.

$$Ans.\ a^{-1}+a^{-2}x+a^{-3}x^2+a^{-4}x^3+a^{-5}x^4+\dots\dots$$

8. Expand $\dfrac{a^5-x^5}{a-x}$ into a series.

$$Ans.\ a^4+a^3x+a^2x^2+ax^3+x^4.$$

9. Expand $\dfrac{bx}{ax+bx^2}$ into a series.

$$Ans.\ \frac{b}{a}-\frac{b^2}{a^2}x+\frac{b^3}{a^3}x^2-\frac{b^4}{a^4}x^3+\dots\dots\dots$$

10. Expand $\dfrac{x}{x+1}$ into a series.

$$Ans.\ x-x^2+x^3-x^4+x^5-x^6+\dots\dots\dots$$

11. Expand $\dfrac{x+1}{x-1}$ into a series.

Ans. $-1-2x-2x^2-2x^3-2x^4-\ldots\ldots$

12. Expand $\dfrac{1+x}{1-x-x^2}$ into a series.

Ans. $1+2x+3x^2+5x^3+8x^4+13x^5+\ldots\ldots$

13. Expand $\dfrac{x}{1+x+x^2}$ into a series.

Ans. $x-x^2+x^4-x^5+x^7-\ldots\ldots$

14. Expand $x(1-x)^{-3}$ into a series.

Ans. $x+3x^2+6x^3+10x^4+\ldots\ldots$

15. Expand $\sqrt{a^2-x^2}$ into a series.

Ans. $a-\dfrac{x^2}{2a}-\dfrac{x^4}{8a^3}-\dfrac{x^6}{16a^5}-\dfrac{5x^8}{128a^7}\ldots\ldots$

16. Expand $\sqrt{64+1}$ into a series.

Ans. $8+\dfrac{1}{16}-\dfrac{1}{8^4}+\dfrac{1}{2\cdot 8^6}\ldots\ldots$

17. Expand $(a+x+x^2)^{\frac{1}{2}}$ into a series.

Ans. $a^{\frac{1}{2}}+\dfrac{x}{2a^{\frac{1}{2}}}+\left(\dfrac{1}{2a^{\frac{1}{2}}}-\dfrac{1}{8a^{\frac{3}{2}}}\right)x^2\ldots\ldots$

18. Expand $(a^2-ax+x^2)^{\frac{1}{2}}$ into a series.

Ans. $a-\dfrac{x}{2}+\dfrac{3x^2}{8a}\ldots\ldots\ldots$

19. Expand $(ax-x^2)^{\frac{1}{2}}$ into a series.

Ans. $a^{\frac{1}{2}}x^{\frac{1}{2}}-\dfrac{x^{\frac{3}{2}}}{2a^{\frac{1}{2}}}+\dfrac{x^{\frac{5}{2}}}{8a^{\frac{3}{2}}}\ldots\ldots\ldots$

20. Expand $a(a^3+b^3)^{-\frac{1}{3}}$ into a series.

Ans. $1-\dfrac{b^3}{3a^3}+\dfrac{2b^6}{9a^6}-\dfrac{14b^9}{81a^9}\ldots\ldots$

21. Expand $\dfrac{1}{3x-x^2}$ into a series.

Ans. $\dfrac{1}{3x}+\dfrac{1}{9}+\dfrac{1}{27}x+\dfrac{1}{81}x^2+\ldots\ldots$

22. Decompose $\dfrac{10x-16}{x^2-4x+3}$. *Ans.* $\dfrac{3}{x-1}+\dfrac{7}{x-3}$.

23. Decompose $\dfrac{11x-37}{x^2-7x+10}$. *Ans.* $\dfrac{5}{x-2}+\dfrac{6}{x-5}$.

24. Decompose $\dfrac{4x-60}{x^2-12x+32}$. *Ans.* $\dfrac{11}{x-4}-\dfrac{7}{x-8}$.

25. Decompose $\frac{6x^2-22x+18}{x^3-6x^2+11x-6}$. *Ans.* $\frac{1}{x-1}+\frac{2}{x-2}+\frac{3}{x-3}$.

26. Decompose $\frac{1}{x^4-1}$ *Ans.* $\frac{1}{4(x-1)}-\frac{1}{4(x+1)}-\frac{1}{2(x^2+1)}$.

27. Decompose $\frac{x+2}{x^3-x}$ *Ans.* $\frac{1}{2(x+1)}+\frac{3}{2(x-1)}-\frac{2}{x}$.

28. Decompose $\frac{x^2}{(x+1)(x+2)(x+3)}$

Ans. $\frac{1}{2(x+1)}-\frac{4}{x+2}+\frac{9}{2(x+3)}$.

29. Decompose $\frac{2x+3}{x^3+x^2-2x}$. *Ans.* $-\frac{3}{2x}-\frac{1}{6(x+2)}+\frac{5}{3(x-1)}$.

30. Decompose $\frac{13+21x+2x^2}{1-5x^2+4x^4}$.

Ans. $\frac{1}{1+x}-\frac{6}{1-x}+\frac{2}{1+2x}+\frac{16}{1-2x}$.

THE METHOD OF INVOLUTION.

PROBLEM.

(418.) Expand $(x+y)^n$ into a series, n being a whole number.

SOLUTION.

This may be expanded by raising $x+y$ to the nth power.

This may be done by actual multiplication, or by the *binomial theorem.* As this theorem is also applicable to evolution, it will be demonstrated in the next method.

THE METHOD OF EVOLUTION.

(419.) A method of extracting roots has already been given. But since the law of the binomial theorem holds good when the exponent is fractional, we have a more expeditious method of extracting the roots of binomials.

BINOMIAL THEOREM.

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{2\ .\ 3}x^{n-3}y^3 + \frac{n(n-1)(n-2)(n-3)}{2\cdot 3\cdot 4}x^{n-4}y^4 + \text{\&c.}$$ x and y being any quantities,

whether real or imaginary, and n being a positive or negative integer, or a positive or negative fraction.

DEMONSTRATION.

This demonstration depends upon the following

LEMMA.

$\left(\dfrac{z^n-u^n}{z-u}\right)_{u=z}=nz^{n-1}$, n being a positive integer or a positive fraction, a negative integer or a negative fraction. The notation after the parenthesis denotes that the first member of the equation equals the second when u is equal to z.

We may divide the proof of this lemma into four cases.

CASE I.

When n is a positive integer.

PROOF.

We have seen in **(112)** that when n is a positive integer that $\dfrac{z^n-u^n}{z-u}=z^{n-1}+z^{n-2}u+z^{n-3}u^2+z^{n-4}u^3\ldots\ldots z^2u^{n-3}+zu^{n-2}+u^{n-1}$.

When $u=z$ each of the terms in the second member becomes z^{n-1}, and since the second member contains n terms, we have $\left(\dfrac{z^n-u^n}{z-u}\right)_{u=z}=nz^{n-1}$, which proves the lemma true in the first case.

CASE II.

When n is a positive fraction.

PROOF.

Supposing $n=\dfrac{p}{q}$, p and q being positive integers, and $z=r^q$, and $u=v^q$, we have

$$\frac{z^{\frac{p}{q}}-u^{\frac{p}{q}}}{z-u}=\frac{r^p-v^p}{r^q-v^q}=\frac{\dfrac{r^p-v^p}{r-v}}{\dfrac{r^q-v^q}{r-v}}$$

According to Case I. $\dfrac{r^p-v^p}{r-v}=pr^{p-1}$, and $\dfrac{r^q-v^q}{r-v}=qr^{q-1}$, when $v=r$.

But since, when $v=r$, u must equal z, we have

$$\left(\frac{z^{\frac{p}{q}}-u^{\frac{p}{q}}}{z-u}\right)_{u=z}=\left(\frac{\dfrac{r^p-q^q}{r-v}}{\dfrac{r^q-v^q}{r-v}}\right)_{v=r}=\frac{pr^{p-1}}{qr^{q-1}}=\frac{p}{q}r^{p-q}$$

But since $r^q=z$, we have $r=z^{\frac{1}{q}}$ and $r^{p-q}=z^{\frac{p-q}{q}}=z^{\frac{p}{q}-1}$. Substituting this value of r^{p-q} in the above expression, we obtain $\left(\frac{z^{\frac{p}{q}}-u^{\frac{p}{q}}}{z-u}\right)_{u=z}=$ $\frac{p}{q}z^{\frac{p}{q}-1}$ which is the formula given in the lemma, when $n=\frac{p}{q}$, and hence proves the lemma to be true in the second case.

CASE III.

When n is a negative integer.

PROOF.

Suppose $n=-m$. Then since $z^{-m}-u^{-m}=-z^{-m}u^{-m}(z^m-u^m)$, we have $\frac{z^{-m}-u^{-m}}{z-u}=-z^{-m}u^{-m}\left(\frac{z^m-u^m}{z-u}\right)$.

But according to Case I., we have $\left(\frac{z^m-u^m}{z-u}\right)_{u=z}=mz^{m-1}$, which being substituted in the above equation, gives $\left(\frac{z^{-m}-u^{-m}}{z-u}\right)_{u=z}=$ $-z^{-m}u^{-m}mz^{m-1}=-mz^{-m-1}$, which is the formula given in the lemma when $n=-m$, and, therefore, proves the lemma to be true in the third case.

CASE IV.

When n is a negative fraction.

PROOF.

The proof in this case is found by putting $-\frac{p}{q}$ for $-m$ in the last, and referring to Case II. instead of Case I.

Therefore, the truth of the lemma is fully established.

We are now ready to proceed with the general demonstration of the binomial theorem. We are required to ascertain the development of $(x+y)^n$.

Since, $(x+y)^n=x^n\left(1+\frac{y}{x}\right)^n$ it is only necessary in order to find the development of $(x+y)^n$ to find that of $\left(1+\frac{y}{x}\right)^n$ and multiply it by x^n.

For convenience put $z=\frac{y}{x}$, and we then have $(1+z)^n$, whose development will now be sought.

Assume $(1+z)^n=A+Bz+Cz^2+Dz^3+Ez^4+\ldots\ldots\ldots$ in which the assumed coefficients A, B, C, D, E, &c., are independent of z, and depend entirely for their values on 1 and n. Let us now endeavor to ascertain the values of these coefficients in terms of 1 and n.

To find the value of A make $z=0$ in the assumed equation, which we have a right to do, since A is independent of z, and we get

$$A=1^n=1.$$

Substituting this value of A in the assumed equation, we have

$$(1+z)^n=1+Bz+Cz^2+Dz^3+Ez^4+\ldots\ldots \quad (A).$$

Since z may be any value whatever without affecting the coefficients A, B, C, &c., let us make $z=u$, and we have

$$(1+u)^n=1+Bu+Cu^2+Du^3+Eu^4+\ldots\ldots \quad (B)$$

Subtracting (B) from (A), we get

$$(1+z)^n-(1+u)^n=B(z-u)+C(z^2-u^2)+D(z^3-u^3)+E(z^4-u^4)+\ldots\ldots$$

Dividing the first member of this equation by $(1+z)-(1+u)$, and the second by its equal $z-u$, we obtain

$$\frac{(1+z)^n-(1+u)^n}{(1+z)-(1+u)}=B+C\left(\frac{z^2-u^2}{z-u}\right)+D\left(\frac{z^3-u^3}{z-u}\right)+E\left(\frac{z^4-u^4}{z-u}\right)+\ldots(C)$$

Putting $1+z=r$ and $1+u=v$ (C) becomes

$$\frac{r^n-v^n}{r-v}=B+C\left(\frac{z^2-u^2}{z-u}\right)+D\left(\frac{z^3-u^3}{z-u}\right)+E\left(\frac{z^4-u^4}{z-u}\right)+\ldots\ldots \quad (D)$$

But by the lemma we have

$$\left(\frac{r^n-v^n}{r-v}\right)_{v=r}=nr^{n-1}.$$

But when $v=r$, we also have $u=z$, and by the

$$\text{Lemma}\quad\left\{\begin{aligned}&\left(\frac{z^2-u^2}{z-u}\right)_{u=z}=2z,\\ &\left(\frac{z^3-u^3}{z-u}\right)_{u=z}=3z^2,\\ &\left(\frac{z^4-u^4}{z-u}\right)_{u=z}=4z^3.\end{aligned}\right.$$

Equation (D) will become by substitution

$$nr^{n-1}=B+2Cz+3Dz^2+4Ez^3.$$

Multiplying this equation by $r=1+z$, we get

$$\begin{aligned}nr^n=B+2Cz+3Dz^2+4Ez^3+\ldots\ldots\ldots\\ +Bz+2Cz^2+3Dz^3+\ldots\ldots\end{aligned}$$

Since $1+z=r$, we have $n(1+z)^n=B+(B+2C)z+(2C+3D)$ $z^2+(3D+4E)z^3+\ldots.$

Multiplying (A) by n, we get

$$n(1+z)^n=n+nBz+nCz^2+nDz^3+\ldots.$$

Whence, $B+(B+2C)z+(2C+3D)z^2+(3D+4E)z^3+\ldots$
$$=n+nBz+nCz^2+nDz^3+\ldots\ldots$$

Equating the coefficients of the like powers,

$$\text{we have}\left\{\begin{array}{c} B=n \\ B+2C=nB, \\ 2C+3D=nC, \\ 3D+4E=nD, \\ \&c. \end{array}\right\},\text{ whence }\left\{\begin{array}{l} B=n, \\ C=\dfrac{n(n-1}{2}, \\ D=\dfrac{n(n-1)(n-2)}{2\ .\ 3}, \\ E=\dfrac{n(n-1)(n-2)(n-3)}{2\ .\ 3\ .\ 4}, \\ \quad\&c. \end{array}\right.$$

Substituting these values of B, C, D, &c., in (A), and restoring the value of $z=\frac{y}{x}$, we get

$$\left(1+\frac{y}{x}\right)^n=1+n\frac{y}{x}+\frac{n(n-1)}{2}\,\frac{y^2}{x^2}+\frac{n(n-1)(n-2)}{2\ .\ 3}\,\frac{y^3}{x^3}$$
$$+\frac{n(n-1)(n-2)(n-3)}{2\ .\ 3\ .\ 4}\,\frac{y^4}{x^4}+\ldots$$

Therefore, since $x^n\left(1+\frac{y}{x}\right)^n=(x+y)^n$, we get by multiplying the expression of $\left(1+\frac{y}{x}\right)^n$ by x^n,

$$(x+y)^n=x^n+nx^{n-1}y+\frac{n(n-1)}{2}x^{n-2}y^2+\frac{n(n-1)(n-2)}{2\ .\ 3}x^{n-3}y^3$$
$$+\frac{n(n-1)(n-2)(n-3)}{2\ .\ 3\ .\ 4}x^{n-4}y^4+\ldots.$$

which is the formula given in the theorem.

For the purpose of reference, we append the following formulas which have been encountered in the above investigation. Which formula it is best to use will depend on the nature of the binomial to be expanded.

$$(x+y)^n=x^n+nx^{n-1}y+\frac{n(n-1)}{2}x^{n-2}y^2+\frac{n(n-1)(n-2)}{2\ .\ 3}x^{n-3}y^3+\ldots \quad (1).$$

$$\left(1+\frac{y}{x}\right)^n=1+n\frac{y}{x}+\frac{n(n-1)}{2}\,\frac{y^2}{x^2}+\frac{n(n-1)(n-2)}{2\ .\ 3}\,\frac{y^3}{x^3}+\ldots \quad (2).$$

$$(1+x)^n=1+nx+\frac{n(n-1)}{2}x^2+\frac{n(n-1)(n-2)}{2\ .\ 3}x^3+\ldots \quad (3).$$

EXAMPLES.

1. Expand $(a^2+x)^{\frac{1}{2}}$ into a series.

$$Ans.\ a+\frac{x}{2a}-\frac{x^2}{2\cdot 4a^3}+\frac{3x^3}{2\cdot 4\cdot 6a^5}-\frac{3\cdot 5x^4}{2\cdot 4\cdot 6\cdot 8a^7}\ \ldots.$$

2. Expand $(a+x)^{\frac{1}{2}}$ into a series.

$$Ans.\ a^{\frac{1}{2}}\left(1+\frac{x}{2a}-\frac{x^2}{2\cdot 4a^2}+\frac{3x^3}{2\cdot 4\cdot 6a^3}-\frac{3\cdot 5x^4}{2\cdot 4\cdot 6\cdot 8a^4}\ \ldots.\right)$$

3. Expand $\sqrt{2}=(1+1)^{\frac{1}{2}}$ into a series.

$$Ans.\ 1+\frac{1}{2}-\frac{1}{2\cdot 4}+\frac{3}{2\cdot 4\cdot 6}-\frac{3\cdot 5}{2\cdot 4\cdot 6\cdot 8}+\ \ldots.$$

4. Expand $(a-b)^{\frac{1}{4}}$ into a series.

$$Ans.\ a^{\frac{1}{4}}\left(1-\frac{b}{4a}-\frac{3b^2}{4\cdot 8a^2}-\frac{3\cdot 7b^3}{4\cdot 8\cdot 12a^3}-\frac{3\cdot 7\cdot 11b^4}{4\cdot 8\cdot 12\cdot 16a^4}\ \ldots.\right)$$

5. Expand $(a+y)^{-4}$ into a series.

$$Ans.\ \frac{1}{a^4}-\frac{4y}{a^5}+\frac{10y^2}{a^6}-\frac{20y^3}{a^7}+\frac{35y^4}{a^8}\ \ldots.$$

6. Expand $\dfrac{c}{a-b}$ into a series.

$$Ans.\ \frac{c}{a}+\frac{bc}{a^2}+\frac{b^2c}{a^3}+\frac{b^3c}{a^4}+\frac{b^4c}{a^5}\ \ldots..$$

7. Expand $(a^2+b^2)^{\frac{1}{2}}$ into a series.

$$Ans.\ a+\frac{b^2}{2a}-\frac{b^4}{8a^3}+\frac{b^6}{16a^5}\ \ldots\ldots\ldots$$

8. Expand $\dfrac{b}{\sqrt{a^2+x^2}}$ into a series.

$$Ans.\ \frac{a}{b}\left(1-\frac{x^2}{2a^2}+\frac{3x^4}{2^3a^4}+\frac{5x^6}{2^4a^6}+\frac{5.7x^8}{2^7a^8}\ \ldots.\right)$$

9. Expand $(c^2-x^2)^{\frac{3}{4}}$ into a series.

$$Ans.\ c^{\frac{3}{2}}\left(1-\frac{3x^2}{2^2c^2}-\frac{3x^4}{2^5c^4}-\frac{5x^6}{2^7c^6}\ \ldots\ldots\ldots\right)$$

10. Find the cube root of $\dfrac{a^3}{a^3+b^3}$.

$$Ans.\ 1-\frac{b^3}{3a^3}+\frac{2b^6}{9a^6}-\frac{14b^9}{81a^9}\ \ldots\ldots$$

11. Expand $\sqrt{5}=\sqrt{4+1}$ into a series.

$$Ans.\ 2+\frac{1}{2^2}-\frac{1}{2^6}+\frac{1}{2^9}-\frac{5}{2^{14}}\ \ldots\ldots$$

12. Expand $\sqrt[3]{6}=\sqrt[3]{8-2}$ into a series.

$$Ans.\ 2-\frac{1}{2\cdot 3}-\frac{1}{2^3\cdot 3^2}+\frac{5}{2^5\cdot 3^4}-\frac{5}{2^6\cdot 3^5}\ \ldots$$

PROPOSITION

(420.) 1. *The sum of the coefficients of the odd terms of the expansion of* $(a+b)^n$ *is equal to the sum of the coefficients of the even terms.*

PROPOSITION

(421.) 2. *The sum of the coefficients in the expansion of* $(a+b)^n$ *is equal to* n*th power of* 2.

PROPOSITION

(422.) 3. *The sum of the coefficients in the expansion of* $(a-b)^n$ *is equal to zero.*

(423.) We have seen that the binomial theorem furnishes the means of expanding polynomials, since all polynomials may become binomials by substitution. But a more direct method of expanding polynomials is by the

MULTINOMIAL THEOREM.

$$(x+ay+by^2+cy^3\ldots)^n=x^n+nx^{n-1}ay+\left(\frac{n(n-1)}{2}x^{n-2}a^2+nx^{n-1}b\right)y^2+$$
$$\left(\frac{n(n-1)(n-2)}{2\ .\ 3}x^{n-3}a^3+n(n-1)x^{n-2}ab+nx^{n-1}c\right)y^3+\ \ldots\ldots\ldots$$

DEMONSTRATION.

Assume $(x+ay+by^2+cy^3\ \ldots\ldots)^{\frac{r}{s}}=A+By+Cy^2+Dy^3+\ldots(A)$.

To determine A make $y=0$. Whence

$$A=x^{\frac{r}{s}},$$

Making $z=y$, we get

$$(x+az+bz^2+cz^3\ \ldots\ldots)^{\frac{r}{s}}=A+Bz+Cz^2+Dz^3+\ \ldots\ldots\ldots(B).$$

Subtracting (B) from (A) there results

$$(x+ay+by^2+cy^3\ \ldots\ldots)^{\frac{r}{s}}-(x+az+bz^2+cz^3\ \ldots)^{\frac{r}{s}}=B(y-z)$$
$$+C(y^2-z^2)+D(y^3-z^3)+\ \ldots(C).$$

Putting $P^s=(x+ay+by^2+cy^3\ \ldots\ldots)$,

whence $P^r=(x+ay+by^2+cy^3\ \ldots\ldots)^{\frac{r}{s}}$,

and $Q^s=(x+az+bz^2+cz^4\ \ldots\ldots)$

whence $Q^r=(x+az+bz^2+cz^3\ \ldots\ldots)^{\frac{r}{s}}$.

we get $P^r-Q^r=B(y-z)+C(y^2-z^2)+D(y^3-z^3)+\ \ldots\ldots$

But $P^s-Q^s=a(y-z)+b(y^2-z^2)+c(y^3-z^3)+\ \ldots\ldots$

Whence, by division, we obtain

$$\frac{P^r-Q^r}{P^s-Q^s}=\frac{B(y-z)+C(y^2-z^2)+D(y^3-z^3)+\;.\;.\;.\;.}{a(y-z)+b(y^2-z^2)+c(y^3-z^3)+\;.\;.\;.\;.}$$

Dividing the numerator and denominator of the first member by $P-Q$ and the numerator and denominator of the second member by $y-z$, we get

$$\frac{P^{r-1}+P^{r-2}Q\;.\;.\;.\;PQ^{r-2}+Q^{r-1}}{P^{s-1}+P^{s-2}Q\;.\;.\;.\;PQ^{s-2}+Q^{s-1}}=$$
$$\frac{B+C(y+z)+D(y^2+yz+z^2)+\;.\;.\;.\;.\;.}{a+b(y+z)+C(y^2+yz+z^2)+\;.\;.\;.\;.\;.}\qquad (D)$$

Now, if we make $y=z$, since P will then equal Q, Eq. (D) becomes

$$\frac{rP^{r-1}}{sP^{s-1}}=\frac{rP^r}{sP^s}=\frac{B+2Cy+3Dy^2+4Ey^3+\;.\;.\;.\;.}{a+2by+3cy^2+4dy^3+\;.\;.\;.\;.}$$

Substituting the values of P^r and P^s, we get

$$\frac{r(x+ay+by^2+cy^3\;.\;.\;.\;.)^{\frac{r}{s}}}{s(x+ay+by^2+cy^3\;.\;.\;.\;.)}=\frac{B+2Cy+3Dy^2+4Ey^3+\;.\;.\;.\;.}{a+2by+3cy^2+4dy^3+\;.\;.\;.\;.}$$

Clearing of fractions, we have

$$\frac{r}{s}(x+ay+by^2+cy^3+dy^4\;.\;.\;.\;)^{\frac{r}{s}}(a+2by+3cy^2+4dy^3+5ey^4\;.\;.\;.\;)=$$
$$(x+ay+by^2+cy^3+dy^4\;.\;.\;.\;)(B+2Cy+3Dy^2+4Ey^3+5Fy^4\;.\;.\;.\;)$$

By substitution from Eq. (A), we get

$$\frac{r}{s}(A+By+Cy^2+Dy^3+Ey^4\ldots)(a+2by+3cy^2+4dy^3+5ey^4\ldots)=$$
$$(x+ay+by^2+cy^3+dy^4\;.\;.\;.\;)(B+2Cy+3Dy^2+4Ey^3+5Fy^4\;.\;.\;.\;)$$

Performing the multiplication indicated, we have

$$\begin{array}{r|r|r|r|r|r|r|l}
\frac{r}{s}aA+aB & \frac{r}{s}y+aC & \frac{r}{s}y^2+aD & \frac{r}{s}y^3+aE & \frac{r}{s}y^4+aF & \frac{r}{s}y^5+aG & \frac{r}{s}y^6+aH & \frac{r}{s}y^7\;.\;.\;. \\
+2bA & +2bB & +2bC & +2bD & +2bE & +2bF & +2bG & \\
 & +3cA & +3cB & +3cC & +3cD & +3cE & +3cF & \\
 & & +4dA & +4dB & +4dC & +4dD & +4dE & \\
 & & & +5eA & +5eB & +5eC & +5eD & \\
 & & & & +6fA & +6fB & +6fC & \\
 & & & & & +7gA & +7gB & \\
 & & & & & & +8hA &
\end{array}$$

[Forward

$$
\begin{array}{r|r|r|r|r|r|r|l}
=Bx+aB & y+bB & y^2+cB & y^3+dB & y^4+eB & y^5+fB & y^6+gB & y^7\ldots \\
+2xC & +2aC & +2bC & +2cC & +2dC & +2eC & +2fC & \\
 & +3xD & +3aD & +3bD & +3cD & +3dD & +3eD & \\
 & & +4xE & +4aE & +4bE & +4cE & +4dE & \\
 & & & +5xF & +5aF & +5bF & +5cF & \\
 & & & & +6xG & +6aG & +6bG & \\
 & & & & & +7xH & +7aH & \\
 & & & & & & +8xI &
\end{array}
$$

Equating the coefficients of the like powers of y, we have

$$Bx=\frac{r}{s}aA$$

$$aB+2xC=aB\frac{r}{s}+2bA\frac{r}{s}$$

$$bB+2aC+3xD=aC\frac{r}{s}+2bB\frac{r}{s}+3cA\frac{r}{s}$$

$$cB+2bC+3aD+4xE=aD\frac{r}{s}+2bC\frac{r}{s}+3cB\frac{r}{s}+4dA\frac{r}{s}$$

$$dB+2cC+3bD+4aE+5xF=aE\frac{r}{s}+2bD\frac{r}{s}+3cC\frac{r}{s}+4dB\frac{r}{s}+5eA\frac{r}{s}$$

$$eB+2dC+3cD+4bE+5aF+6xG=aF\frac{r}{s}+2bE\frac{r}{s}+3cD\frac{r}{s}+4dC\frac{r}{s}+5eB\frac{r}{s}+6fA\frac{r}{s}$$

There is a symmetry about these equations which would enable us to form successive ones without resorting to the equation from which these have been derived.

Putting $n=\frac{r}{s}$, and instead of A write its value $x^{\frac{r}{s}}=x^n$, and solving the above equations, we have

$A=x^n$

$B=nax^{n-1}$

$C=\frac{n(n-1)}{2}a^2x^{n-2}+nbx^{n-1}$

$D=\frac{n(n-1)(n-2)}{2\ .\ 3}a^3x^{n-3}+n(n-1)abx^{n-2}+ncx^{n-1}$

$E=\frac{n(n-1)(n-2)(n-3)}{2\ .\ 3\ .\ 4}a^4x^{n-4}+\frac{n(n-1)(n-2)}{2}a^2bx^{n-3}+\frac{n(n-1)}{2}(2ac+b^2)x^{n-2}$
$+ndx^{n-1}$.

It may be seen from these values we are not able to write all the terms in the value of F, although some are apparent. But, by using A, B, C, D, &c., instead of their values, the law becomes evident.

Thus, we may obtain the following general formula:

$$
\begin{aligned}
&(x+ay+by^2+cy^3+dy^4+ey^5+fy^6+gy^7+hy^8+\ldots\ldots\ldots\ldots\ldots)^n=\\
&x^n\\
&+naAx^{-1}y\\
&+\tfrac{1}{2}\Big((n-1)aB+2nbA\Big)x^{-1}y^2\\
&+\tfrac{1}{3}\Big((n-2)aC+(2n-1)bB+3ncA\Big)x^{-1}y^3\\
&+\tfrac{1}{4}\Big((n-3)aD+(2n-2)bC+(3n-1)cB+4ndA\Big)x^{-1}y^4\\
&+\tfrac{1}{5}\Big((n-4)aE+(2n-3)bD+(3n-2)cC+(4n-1)dB+5neA\Big)x^{-1}y^5\\
&+\tfrac{1}{6}\Big((n-5)aF+(2n-4)bE+(3n-3)cD+(4n-2)dC+(5n-1)eB+6nfA\Big)x^{-1}y^6\\
&+\tfrac{1}{7}\Big((n-6)aG+(2n-5)bF+(3n-4)cE+(4n-3)dD+(5n-2)eC+(6n-1)fB+7ngA\Big)x^{-1}y^7\\
&+\tfrac{1}{8}\Big((n-7)aH+(2n-6)bG+(3n-5)cF+(4n-4)dE+(5n-3)eD+(6n-2)fC+(7n-1)gB+8nhA\Big)x^{-1}y^8\\
&+\tfrac{1}{9}\Big((n-8)aI+(2n-7)bH+(3n-6)cG+(4n-5)dF+(5n-4)eE+(6n-3)fD+(7n-2)gC+(8n-1)hB+9niA\Big)x^{-1}y^9\\
&+\tfrac{1}{10}\Big((n-9)aJ+(2n-8)bI+(3n-7)cH+(4n-6)dG+(5n-5)eF+(6n-4)fE+(7n-3)gD+(8n-2)hC+(9n-1)iB+10njA\Big)x^{-1}y^{10}
\end{aligned}
$$

REMARK.—This theorem applies, of course, to both involution and evolution. The student should use the multinomial theorem in solving the following

EXAMPLES.

1. Expand $(1+x+x^2+x^3+x^4\ \ldots)^3$ into a series.

Ans. $1+3x+6x^2+10x^3+15x^4+\ \ldots\ldots$

2. Expand $(2x+3x^2+4x^3\ \ldots\ldots)^3$ into a series.

Ans. $8x^3+36x^4+102x^5+231x^6+\ \ldots\ldots$

3. Expand $(1+x+x^2+x^3+x^4\ \ldots\ldots)^{\frac{1}{2}}$ into a series.

Ans. $1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\frac{35}{128}x^4+\ \ldots\ldots$

4. Expand $(1+\frac{1}{2}x+\frac{1}{3}x^2+\frac{1}{4}x^3\ \ldots)^{\frac{1}{3}}$ into a series.

Ans. $1+\frac{1}{6}x+\frac{1}{12}x^2+\frac{35}{648}x^3+\frac{383}{9720}x^4+\ \ldots\ldots$

REVERSION OF SERIES.

(424.) The *reversion of a series* is finding another series which is equal to the unknown quantity in the given series.

PROBLEM

(425.) 1. Revert the series $ax+bx^2+cx^3+dx^4\ \ldots\ldots$

SOLUTION.

Assume that $x=Ay+By^2+Cy^3+Dy^4\ \ldots\ldots$, y being equal to $ax+bx^2+cx^3+dx^4\ \ldots$

Substituting the value x in the equation

$$y=ax+bx^2+cx^3+dx^4\ \ldots,$$

we have

$$y+0y^2+0y^3+0y^4=aAy+\begin{array}{r|}aB\\ bA^2\end{array}\,y^2+\begin{array}{r|}aC\\ +2bAB\\ cA^3\end{array}\,y^3+\begin{array}{r|}aD\\ +\ 2bAC\\ +\ \ bB^2\\ +3cA^2B\\ dA^4\end{array}\,y^4+\ \ldots$$

Equating the like coefficients of y,

we have
$$\left.\begin{array}{r}aA=1\\ bA^2+aB=0\\ cA^3+2bAB+aC=0\\ dA^4+3cA^2B+bB^2+2bAC+aD=0\end{array}\right\},\ \text{whence}\ \left\{\begin{array}{l}A=\dfrac{1}{a},\\ B=-\dfrac{b}{a^3}\\ C=\dfrac{2b^2-ac}{a^5},\\ B=-\dfrac{5b^3-5abc+a^2d}{a^7}.\end{array}\right.$$

Therefore, $x=\dfrac{1}{a}y-\dfrac{b}{a^3}y^2+\dfrac{2b^2-ac}{a^5}y^3-\dfrac{5b^3-5abc+a^2d}{a^7}y^4\ \ldots$

PROBLEM

2. Revert $y=ax+bx^3+cx^5+dx^7\ \dots$

SOLUTION.

Assume $x=Ay+By^3+Cy^5+Dy^7\ \dots$

Substituting x for its value, we have

$$y+0y^3+0y^5+0^7=aAy+\left.\begin{array}{r}aB\\ bA^3\end{array}\right|y^3+\left.\begin{array}{r}aC\\ +3bA^2B\\ cA^5\end{array}\right|y^5+\left.\begin{array}{r}aD\\ +3bA^2C\\ +5cA^4B\\ dA^7\\ +3bAB^2\end{array}\right|y^7+\dots$$

Equating like coefficients, we obtain

$$\left.\begin{array}{r}aA=1\\ bA^3+aB=0\\ cA^5+3bA^2B+aC=0\\ 3bAB^2+dA^7+5cA^4B+3bA^2C+aD=0\end{array}\right\}\text{, whence }\left\{\begin{array}{l}A=\dfrac{1}{a},\\ B=-\dfrac{b}{a^4},\\ C=\dfrac{3b^2-ac}{a^7},\\ D=-\dfrac{12b^3-8abc+a^2d}{a^{10}}.\end{array}\right.$$

PROBLEM

3. Revert $ax+bx^2+cx^3+dx^4\ \dots=py+qy^2+ry^3+sy^4\ \dots$

SOLUTION.

Assume $y=Ax+Bx^2+Cx^3+Dx^4\ \dots$

Substituting this value of y, we have

$$ax+bx^2+cx^3+dx^4\dots=pAx+\left.\begin{array}{r}pB\\ qA^2\end{array}\right|x^2+\left.\begin{array}{r}pC\\ +2qAB\\ rA^3\end{array}\right|x^3+\left.\begin{array}{r}pD\\ +\ 2qAC\\ +\ qB^2\\ +3rA^2B\\ sA^4\end{array}\right|x^4$$

Equating the like coefficients of x, there results

$$\left.\begin{array}{r}pA=a\\ qA^2+pB=b\\ rA^3+2qAB+pC=c\\ sA^4+3rA^2B+qB^2+2qAC+pD=d\end{array}\right\}\text{, whence }\left\{\begin{array}{l}A=\dfrac{a}{p},\\ B=\dfrac{bp^2-a^2q}{p^3},\\ C=\dfrac{cp^4-2abp^2q+a^3q+2a^3pr}{p^5}.\\ D=\dfrac{cp^4-a^4pr+2aq(a^2q-bp^2)}{p^5}.\end{array}\right.$$

Therefore, $y=\dfrac{a}{p}x+\dfrac{bp^2-a^2q}{p^3}x^2+\dfrac{cp^4-a^4pr+2aq(a^2q-bp^2)}{p^5}x^3+\dots$

PROBLEM

4. Revert $p+ax+bx+cx^2+dx^4\ldots=y$.

SOLUTION.

Transposing p, we have

$$ax+bx+cx^2+dx^4\ \ldots=y-p.$$

This series is the same as the one in the first problem except that $y-p$ stands in the place of y. Therefore, its reversion must be.

$$x=\frac{1}{a}(y-p)-\frac{b}{a^3}(y-p)^2+\frac{2b^2-ac}{a^5}(y-p)^3$$
$$-\frac{5b^3-5abc+a^2d}{a^7}(y-p)^4\ldots\ldots$$

EXAMPLES.

1. Revert $y=x+x^2+x^3+x^4\ \ldots$

Ans. $x=y-y^2+y^3-x^4+\ldots\ldots\ldots$

2. Revert $y=x-\frac{1}{2}x^2+\frac{1}{4}x^3-\frac{1}{8}x^4\ \ldots$

Ans. $x=y+\frac{1}{2}y^2+\frac{1}{4}y^3+\frac{1}{8}y^4\ \ldots\ldots\ldots$

3. Revert $y=x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7\ \ldots$

Ans. $x=y+\frac{1}{3}y^3+\frac{2}{15}y^5+\frac{17}{315}y^7\ \ldots\ldots$

4. Revert $y=2x+3x^3+4x^5+5x^7\ \ldots$

Ans. $x=\frac{1}{2}y-\frac{3}{16}y^3+\frac{19}{128}y^5-\frac{152}{1024}y^7\ \ldots$

5. Revert $y=1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4\ \ldots$

Ans. $x=y-1-\frac{1}{2}(y-1)^2+\frac{1}{3}(y-1)^3-\dfrac{(y-1)^4}{4}\ldots$

6. Revert $y=x+2x^2+4x^3\ \ldots$

Ans. $x=y-2y^2-4y^3\ldots\ldots$

7. Revert $y=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\frac{1}{5}x^5\ \ldots$

Ans, $x=y+\dfrac{y^2}{2}+\dfrac{y^3}{2\cdot3}+\dfrac{y^4}{2\cdot3\cdot4}+\dfrac{y^5}{2\cdot3\cdot4\cdot5}\ \ldots$

8. Revert $y=x-\dfrac{x^3}{2\cdot3}+\dfrac{x^5}{2\cdot3\cdot4\cdot5}-\dfrac{x^7}{2\cdot3\cdot4\cdot5\cdot6\cdot7}\ \ldots$

Ans. $x=y+\dfrac{x^3}{2\cdot3}+\dfrac{1\cdot3x^5}{2\cdot4\cdot5}+\dfrac{1\cdot3\cdot5x^7}{2\cdot4\cdot6\cdot7}+\ldots\ldots$

9. Revert $y=x+3x^2+5x^3+7x^4+9x^5\ \ldots\ldots$

Ans. $x=y-3y^2+13y^2-53y^4\ \ldots\ldots$

10. Revert $y=x+\dfrac{x^2}{2}+\dfrac{x^3}{3}+\dfrac{x^4}{4}+\dfrac{x^5}{5}\ \ldots$

Ans. $x=y-\dfrac{y^2}{2}+\dfrac{y^3}{2\cdot3}-\dfrac{y^4}{2\cdot3\cdot4}\ \ldots\ldots$

SUMMATION OF SERIES.

The summation of a series depends upon its nature.

ARITHMETICAL SERIES.

(426.) The summation of a finite arithmetical series, whether increasing or decreasing, we have already seen is embraced in the formula $S=\left(\frac{a+l}{2}\right)n$, in which a is the first term, l the last term, and n the number of terms.

GEOMETRICAL SERIES.

The summation of a finite geometrical series, whether increasing or decreasing, is embraced in the formula $S=\frac{lr-a}{r-1}$ or $\frac{a-lr}{1-r}$.

When the series is infinite and decreasing, the formula is $S=\frac{a}{1-r}$.

RECURRING SERIES.

A *recurring series* is one in which a certain number of consecutive terms, taken in any part of the series, has a fixed relation to the term which immediately follows.

Thus, in the series $1+3x+4x^2+7x^3+11x^4+18x^5$ . . . . the sum of the coefficients of any two consecutive terms is equal to the coefficient of the following term.

In the series $1+2x+3x^2+4x^3+5x^4+6x^5$ . . . . each term after the second is equal to the product of $2x$ into the first preceding term plus the product of $-x^2$ into the second preceding term.

Thus $5x^2=2x\times 4x^3-x^2\times 3x^2$. The coefficients of x and x^2, or $2-1$, is called the *scale of relation.*

In the series $1+4x+6x^2+11x^3+28x^4+63x^5$ . . . . . , $2-1+3$ is the scale of relation. Thus $63x^5=2x\times 28x^4-x^2\times 11x^3+3x^3\times 6x^2$.

(427.) The fraction $\frac{a}{b+cx}$ produces the series

$$\frac{a}{b}-\frac{acx}{b^2}+\frac{ac^2x^2}{b^3}-\frac{ac^3x^4}{b^4} \;.\;.\;.\;.\;.\;.\;.$$

in which each term after the first is equal to the product of $-\frac{cx}{b}$ into

the term that precedes it. In this series, $-\frac{cx}{b}$ is the *scale of relation* of the terms, and $\frac{c}{b}$ is the *scale of relations* of the coefficients.

(428.) The fraction $\frac{a+bx}{c+dx+ex^2}$ produces the series.

$$\frac{a}{c}+\left(\frac{b}{c}-\frac{ad}{c^2}\right)x-\left(\frac{bd}{c^2}-\frac{ad^2}{c^3}+\frac{ae}{c^2}\right)x^2 \ldots\ldots$$

in which each term, commencing at the third, is equal to the two immediately preceding multiplied respectively by $-\frac{ex^2}{c}$, $-\frac{dx}{c}$, which is the scale of relation of the terms.

(429.) The fraction $\frac{a+bx+cx^2}{d+ex+fx^2+gx^3}$, will produce a series in which each term commencing at the fourth is equal to the three preceding terms multiplied respectively by $-\frac{gx^3}{d}$, $-\frac{fx^2}{d}$, $-\frac{ex}{d}$, which is therefore the *scale of relation* of the terms.

(430.) The fraction $\frac{a+bx+cx^2 \ldots px^n}{q+rx+sx^2 \ldots ux^n+vx^{n+1}}$ will produce a series in which each term commencing at the $n+2$th will be equal to the $n+1$ preceding terms multiplied respectively by

$$-\frac{vx^{n+1}}{q}, -\frac{ux^n}{q}, \ldots\ldots -\frac{sx^2}{q}, -\frac{rx}{q},$$

which is the *scale of relation* of the terms.

When $q=1$ the scale of relation is $-vx^{n+1}$, $-ux^n$, ... $-sx^2$, $-rx$, which are the terms of the denominator taken in reverse order and with contrary signs, the first term being omitted.

(431.) A recurring series is said to be of the *first order* when each term commencing with the second depends upon the one that precedes it.

(432.) A recurring series is said to be of the *second order* when each term commencing with the third depends on the two preceding terms.

In like manner we have recurring series of the *third order*, *fourth order*, &c.

PROBLEM

(433.) 1. To find the scale of relation in a recurring series of the first order.

SOLUTION.

Let A, B, C, D, &c., . . . be the coefficients of a recurring series of the first order.

Now, some relation exists between each two consecutive terms. Let r be the relation. and we have $B=rA$; $C=rB$; $D=rC$, &c.

$$\text{whence,}\quad r=\frac{B}{A}=\frac{C}{B}=\frac{D}{C},\ \text{\&c.}$$

PROBLEM

(434.) 2. To find the scale of relation in a recurring series of the second order.

SOLUTION.

Let A, B, C, D, &c., be the coefficients of the series. Whence by the conditions must arise the following equations.

$$C=mA+nB,$$
$$D=mB+nC,$$
$$\text{\&c.,}\qquad \text{\&c.,}$$

in which m and n are unknown quantities. The solution of these two equations must give their values.

$$BC=mAB+nB^2,$$
$$AD=mAB+nAC,$$
$$AD-BC=(AC-B^2)n,$$
$$n=\frac{AD-BC}{AC-B^2},$$

In the same way m may be found, whose value is seen in the equation

$$m=\frac{C^2-BD}{AC-B^2}.$$

PROBLEM

(435.) 3. To find the scale of relation in a recurring series of the third order.

SOLUTION.

Let A, B, C, D, E, F, &c., be the coefficients. By the conditions, we must have

$$D=mA+nB+qC,$$
$$E=mB+nC+qD,$$
$$F=mC+nD+qE,$$
$$\text{\&c.,}\qquad \text{\&c.}$$

The solution of these equations gives the values of m, n, and q.

PROBLEM

(436.) 4. To find whether the law of the series depends on *one*, two, three, or more terms.

SOLUTION

See whether the scale of relation established by problem 1st agrees with the given series; if not, try problem 2d; and if that does not agree, try problem 3d, and so on until the scale is found. Or we assume that the series is of a higher order than necessary, in which case one or more of the values found by resolving the resulting equations will be found to be zero, and the remaining one will give the true scale of relation.

PROBLEM

(437.) 5. To find the sum of a recurring series of the first order.

SOLUTION.

It is evident that when the series is of the first order (that is, a geometrical series), the sum of the series may be obtained from the formulas, $S=\frac{lr-a}{r-1}$, $S=\frac{a-lr}{1-r}$, and $S=\frac{a}{1-r}$.

The first formula must be used when the series is increasing and finite; the second, when it is decreasing and finite; and the third, when it is decreasing and infinite.

PROBLEM

(438.) 6. To find the sum of an infinite recurring series of the second order.

SOLUTION.

Let $A+B+C+D$ .... be a series of the second order.

$$\begin{aligned} \text{Then } C&=mAx^2+nBx \\ D&=mBx^2+nCx \\ &\vdots \\ R&=mPx^2+nQx \\ &\vdots \end{aligned}$$

Adding these equations together, and putting $A+B+C+D \dots R \dots = S$, we shall have

$$S-A-B=mSx^2+n(S-A)x$$
$$(1-nx-mx^2)S=A+B-nAx$$
$$S=\frac{A+B-nAx}{1-nx-mx^2}$$

PROBLEM

(439.) 7. To find the sum of an infinite recurring series of the third order.

SOLUTION.

Let $A+B+C+D$, &c. be the series.

According to the nature of a recurring series of the third order there must result the following equation:

$$D=mAx^3+nBx^2+qCx$$
$$E=mBx^3+nCx^2+qDx$$
$$\vdots \quad \vdots \quad \vdots$$

Adding these equations, and putting $S=A+B+C+D+E$, &c., we have $S-A-B-C=mSx^3+n(S-A)x^2+q(S-A-B)x$

whence, $$S=\frac{A+B+C-(A+B)qx-nAx^2}{1-qx-nx^2-mx^3}$$

PROBLEM

(440.) 8. To find the sum of a given number of terms of a recurring series of the second order.

SOLUTION.

Let $A+B+C+D \ldots . +R$ be a finite recurring series of the second order. We have

$$C=mAx^2+nBx$$
$$D=mBx^2+nCx$$
$$\vdots$$
$$T=mRx^2+nSx.$$

Let us find the sum of the recurring series of the second order, $U+V+W$, &c., to infinity.

We have, according to Prob. 6,

$$S^1=\frac{U+V-nUx}{1-nx-mx^2}$$

Supposing $A+B+C$, &c., to be an infinite series, we have already found its sum to be

$$\frac{A+B-nAx}{1-nx-mx^2}$$

Subtracting from this the sum of $U+V+W$, &c., to infinity, and there must remain the sum of the finite series, $A+B+C \ldots . T$. Putting this sum equal to S, we have

$$S=\frac{A+B-U-V-nAx+nUx}{1-nx-mx^2}$$

NOTE.—In the same way we might obtain the sum of a finite recurring series of higher orders.

PROBLEM

(441.) 9. To find the general term of a recurring series.

SOLUTION.

The general generating fraction, $\frac{a+bx+cx^2 \ldots . px^n}{q+rx+sx^2 \ldots . +vx^{n+1}}$, may be thus represented :

$$(a+bx+cx^2 \ldots . px^n)(q+rx+sx^2 \ldots . +vx^{n+1})^{-1}$$

The second parenthesis may be expanded by the *multinomial theorem*, and the result multiplied by the quantity in the first parenthesis. Then, if we take in this product the part which contains x to any power whatsoever, we shall obtain the general term of the recurring series.

We present another method.

Take the generating fraction,

$$\frac{a+bx+cx^2 \ldots px^n}{q+rx+sx^2 \ldots vx^{n+1}}$$

which is supposed to be reduced to its lowest terms.

Dividing both numerator and denominator by v, and changing the order of their terms, we have

$$\frac{\frac{p}{v}x^n \ldots \ldots \frac{c}{v}x^2+\frac{b}{v}x+\frac{a}{v}}{x^{n+1} \ldots \ldots \frac{s}{v}x^2+\frac{r}{v}x+\frac{q}{v}}$$

Separate the denominator into binomial factors. Then the fraction may be considered as made up of fractions which have these binomial

fractions for their denominators. Determine these partial fractions according to the method given in undetermined coefficients, and then find the general term of development of each of these partial fractions, and the sum of these general terms will be the general term of the recurring series.

NOTE.—In the decomposition into partial fractions, when a factor of the denominator is of the form $(x+a)(x+b)^m$, assume the fraction to be equal:

$$\frac{A}{x+a}+\frac{B}{(x+b)^m}+\frac{C}{(x+b)^{m-1}}\cdots\frac{H}{x+b}$$

Each partial fraction may be put under the form $P(p+x)^{-r}$ in which r is some positive integral number.

The expansion of this by the binomial theorem gives for the term which contains x^n the following

$$\frac{-r(-r-1)(-r-2)\ldots.(-r-(n-1)}{1\cdot2\cdot3\quad\ldots.n}Pp^{-r-n}x^n.$$

It is the sum of the terms containing x^n resulting from the different partial fractions, that composes the general term in the recurring series.

(442.) A simpler mode of finding the general term of a recurring series will be found in the following discussion.

We have already seen that a recurring series of the first order is a geometrical series, and, therefore, its general term is easily found.

Let us suppose that $a+ar+ar^2+ar^3+ar^4\ldots$, a geometrical series, is also a recurring series of the second order.

When this is the case, we must then have the following equations:

$$\begin{aligned}ar^2&=ma+nar,\\ ar^3&=mar+nar^2,\\ ar^4&=mar^2+nar^3,\\ &\vdots\\ ar^s&=mar^{s-2}+nar^{s-1},\end{aligned}$$

in which m, n is the scale of relation.

By division each of the above equations reduces to

$$r^2=m+nr,$$

whence $$r=\frac{n\pm\sqrt{4m+n^2}}{2}.$$

If these two values of r are not equal, we see that there may be two geometrical series which will also be recurring series of the second order, having m, n for their scale of relation.

Let the two values of r be c, and b. Then since the first terms of these two geometrical recurring series of the second order, may be any quantities whatever; let the variables x and y represent them. We shall then have

$$x+xc+xc^2+xc^3 \ldots\ldots xc^{n-1},$$
$$y+yb+yb^2+yb^3 \ldots\ldots yb^{n-1}.$$

Each of these series are recurring and of the second order, and the scale of relation in each is m, n.

By adding them, we have

$$(x+y)+(cx+by)+(c^2x+b^2y)+(c^3x+b^3y) \ldots\ldots (xc^{n-1}+yb^{n-1}).$$

This series is not geometrical, although formed by the addition of two series which are geometrical. But these two series are also recurring series of the second order. Have we a right then to conclude because two geometrical series added together, do not produce a geometrical series, that these two geometrical recurring series have not when added produced a recurring series?

Let us examine. If the series

$$(x+y)+(cx+by)+(c^2x+b^2y) \ldots.$$

is a recurring series of the second order, whose scale of relation is m, n, we must have

$$c^2x+b^2y=m(x+y)+n(cx+by)$$

Since $x+cx+c^2x+c^3x$, &c., and $y+by+b^2y+b^3y$, &c., are recurring series whose scale of relation is m, n, we have the following equations:

$$c^2x=mx+ncx$$
$$b^2y=my+nby.$$

By addition $c^2x+b^2y=m(x+y)+n(cx+by)$ which proves that

$$(x+y)+(ax+by)+(c^2x+b^2y) \ldots. (xa^{n-1}+yb^{n-1})$$

is a recurring series of the second order, whose scale of relation is m, n.

Let this series be represented by

$A+B+C+D \ldots\ldots T$, T standing for the general term. The following equations must then subsist:

$$x+y=A,$$
$$cx+by=B,$$

whence,
$$x=\frac{B-bA}{c-b},$$
$$y=-\frac{B-cA}{c-b}.$$

We also have $T=xc^{n-1}+yb^{n-1}$.

$$\therefore \quad T=\frac{B-bA}{c-b}c^{n-1}-\frac{B-cA}{c-b}b^{n-1}.$$

By assuming a geometrical series to be a recurring series of the third order, we should find r by solving a cubic equation. Then proceeding in a similar manner, we should find that a recurring series of the third order is equal to the sum of three geometrical series, having the same scale of relation, whence the general term for a series of the third order might be obtained.

EXAMPLES

1. Find the sum of $1+2x+8x^2+28x^3+100x^4+356x^5$,&c.

Ans. Scale $2x^2$, $3x$; sum $\dfrac{1-x}{1-3x-2x^2}$.

2. Find the sum of $1+2x+3x^2+4x^3+5x^4$, &c.

Ans. Scale $-x$, $2x$; sum $\dfrac{1}{1-2x+x^2}$.

3. Find the sum of $1+3x+4x^2+7x^3+11x^4+18x^5+29x^6$, &c.

Ans. Scale x^2, x; sum $\dfrac{1+2x}{1-x-x^2}$.

4. Find the sum of $1+x+5x^2+13x^3+41x^4+121x^5+365x^6$ &c.

Ans. Scale $3x^2$, $2x$; sum $\dfrac{1-x}{1-2x-3x^2}$.

5. Find the sum of $1+6x+12x^2+48x^3+120x^4$, &c.

Ans. Scale $6x^2$, x; sum $\dfrac{1+5x}{1-x-6x^2}$.

6. Find the sum of $1+3x+5x^2+7x^3+9x^4+11x^5$, &c.

Ans. Scale $-x^2$, $2x$; sum $\dfrac{1+x}{1-2x+x^2}$.

7. Find the sum of $1+3x+2x^2-x^3-3x^4-2x^5+x^6$, &c.

Ans. Scale $-x^2$, x; sum $\dfrac{1+2x}{1-x+x^2}$.

8. Find the sum of $\dfrac{a}{b}-\dfrac{ac}{b^2}x+\dfrac{ac^2}{b^3}x^2-\dfrac{ac^3}{b^4}x^3$, &c.

Ans. Scale $-\dfrac{c}{b}x$; sum $\dfrac{a}{b+cx}$.

9. Find the sum of $3+5x+7x^2+13x^3+23x^4$, &c.

Ans. Scale $-2x^3$, x^2, $2x$; sum $\dfrac{3-x-6x^2}{1-2x-x^2+2x^3}$.

10. Find the sum of $1+x+2x^2+2x^3+3x^4+3x^5+4x^6+4x^7$, &c.

Ans. Scale $-x^3$, x^2, x; sum $\dfrac{1}{1-x-x^2+x^3}$.

11. Find the sum of $1+4x+6x^2+11x^3+28x^4+63x^5$, &c.

Ans. Scale $3x^3, -x^2, 2x$; sum $\dfrac{(1+x)^2-2x^2}{(1-x)^2-3x^3}$.

12. Find the nth term of $1+2x+3x^2+4x^3+5x^4$, &c.

Ans. nx^{n-1}.

13. Find the nth term of $1+3x+5x^2+7x^3+9x^4$, &c.

Ans. $(2n-1)x^{n-1}$.

14. Find the nth term of $1+x+5x^2+13x^3+41x^4+121x^5$, &c.

Ans. $\frac{1}{2}(3x)^{n-1}+\frac{1}{2}(-x)^{n-1}$.

15. Find the nth term of $1+x+3x^2+5x^3+11x^4+21x^5$, &c.

Ans. $\frac{2}{3}(2x)^{n-1}+\frac{1}{3}(-x)^{n-1}$.

16. Find the sum of n terms of $1+2x+3x^2$ . . . . .

Ans. $\dfrac{1-(n+1)x^n+nx^{n+1}}{1-2x+x^2}$.

17. Find the sum of n terms of $1+3x+5x^2+7x^3$ . . . . .

Ans. $\dfrac{1+x-(2n+1)x^n+(2n-1)x^{n+1}}{1-2x+x^2}$.

18. Find two geometrical recurring series, which when added shall be equal to the series $1+x+5x^2+13x^3+41x^4$ . . .

Ans. $\begin{cases} \frac{1}{2}+1\frac{1}{2}x+4\frac{1}{2}x^2+13\frac{1}{2}x^3+40\frac{1}{2}x^4 \ldots \\ \dfrac{1}{2}-\frac{1}{2}x+\frac{1}{2}x^2-\frac{1}{2}x^3+\frac{1}{2}x^4 \ldots \end{cases}$

19. Find two geometrical recurring series whose sum shall be equal to the series $1+x+3x^2+5x^3+11x^4$ . . . .

Ans. $\begin{cases} \frac{2}{3}+1\frac{1}{3}x+2\frac{2}{3}x^2+5\frac{1}{3}x^3+10\frac{2}{3}x^4 \ldots \\ \frac{1}{3}-\frac{1}{3}x+\frac{1}{3}x^2-\frac{1}{3}x^3+\frac{1}{3}x^4 \ldots \end{cases}$

20. Find three geometrical recurring series whose sum shall be equal to the series $1+x+2x^2+3x^3+6x^4$ . . . .

Ans. $\begin{cases} \frac{1}{2}+\frac{1}{2}x+\frac{1}{2}x^2+\frac{1}{2}x^3+\frac{1}{2}x^4 \ldots. \\ \frac{1}{3}+\frac{2}{3}x+\frac{4}{3}x^2+\frac{8}{3}x^3+\frac{16}{3}x^4 \ldots. \\ \frac{1}{6}-\frac{1}{6}x+\frac{1}{6}x^2-\frac{1}{6}x^3+\frac{1}{6}x^4 \ldots. \end{cases}$

(443.) An *increasing series* is one whose successive terms increase in value.

(444.) A *decreasing series* is one whose successive terms decrease in value.

(445.) A *converging series* is one in which the greater the number of terms taken, the nearer will their sum approximate the true sum of the series.

PROBLEM.

(446.) To find the degree of approximation when a limited number of terms are taken in a decreasing converging series whose terms are alternately positive and negative.

SOLUTION.

Let $a-b+c-d+\ .\ .\ .\ .\ .\ .\ .\ +m-n+p-q+r-\ .\ .\ .\ .\ .$ represent a decreasing converging series, whose odd terms are positive, and even terms negative.

Since a is greater than b, c than d, m than n, it follows that $a-b$, $c-d$, $m-n$, $p-q$, &c., are positive quantities.

CASE I.

When the number of terms taken is even.

Let $a-b+c-d+\ .\ .\ .\ .\ .\ m-n+p-q$ be the series to n terms, n being even.

Since $r-s$, $l-u$, and the other rejected couplets are each positive, it follows that the above series to the n terms, when n is even, is less than the true sum of the whole series. Hence, we have the following inequation, $a-b+c-d\ .\ .\ .\ .\ .\ .\ m-n+p-q<S$, in which S represents the *true* sum of the whole series.

Now let $a-b+c-d\ .\ .\ .\ .\ .\ .\ m-n+p-q+r$ be the series to $n+1$ terms, $n+1$ being an odd number.

The terms after r are $-s+t-u+v\ .\ .\ .\ .\ .\ .$ These terms may be thus represented, $-(s-t)-(u-v)-\ .\ .\ .\ .$ The quantities in the parenthesis are positive, and since each parenthesis is negative, it follows that the series which terminates in r is followed by a series of negative quantities, whence the following inequation, $a-b+c-d\ .\ .\ .\ .\ .\ .\ m-n+p-q+r>S$.

From this, we see that the quantity necessary to be added to the first member of the first inequation to make it equal to S must be less than r.

CASE II.

When the number of terms taken is odd.

Let $a-b+c-d\ .\ .\ .\ .\ .\ .\ .\ m-n+p$ be the series to n terms, n being odd.

Reasoning, as in the last case, we have the following inequations,

$$a-b+c-d\ .\ .\ .\ .\ .\ m-n+p>S,$$

$$\text{and}\quad a-b+c-d\ .\ .\ .\ .\ .\ m-n+p-q<S,$$

which show that the quantity to be subtracted from the first member of the first inequation to make it equal to S must be less than q.

The solution of this problem establishes the following

THEOREM.

(447.) *The error committed by taking n terms of a decreasing converging series, whose terms are alternately positive and negative, for the sum of the whole series, is numerically less than the following term.*

NOTE.—In applying this theorem to certain examples, the series should be considered as commencing at the second term of the expansion.

THE DIFFERENTIAL METHOD.

(448.) The *differential method* of summing a series is a method of finding the sum of a series by ascertaining the successive differences of its terms.

PROBLEM.

(449.) 1. To find the first term of any order of differences.

SOLUTION.

Let a, b, c, d, e, f, &c., be a series.

Supposing this series to be ascending, we have a new series by taking the first term from the second, the second from the third, and so on. This series is called the *first order of differences.*

In like manner we may take the differences of the successive terms of this new series, and thus form still another series, which is called the second order of differences. By continuing the process, we should obtain other orders of differences. Thus, we may have

1st order $b-a,\ c-b,\ d-c,\ e-d,\ f-e$ . . . .
2d order $c-2b+a,\ d-2c+b,\ e-2d+c,\ f-2e+d$ . . . .
3d order $d-3c+3b-a,\ e-3d+3c-b,\ f-3e+3d-c$ . . . .
4th order $e-4d+6c-4b+a,\ f-4e+6d-4c+b$ . . . . .
5th order $f-5e+10d-10c+5b-a$ . . . .
&c., &c., &c.

By inspecting the coefficients of the terms in the successive order of differences, we see that in the first order they are the same as the coefficients of the expansion of $(x-y)^1$; in the second order, the same as the coefficients of the expansion of $(x-y)^2$; in the third order, as those in the expansion of $(x-y)^3$; and in the nth order, the coefficients must be the same as those in the expansion of $(x-y)^n$. The first in each of the above order of differences may be written thus;

1st order $-(a-b)$.
2d order $a-2b+c$.
3d order $-(a-3b+3c-d)$.
4th order $a-4b+6c-4d+e$.
5th order $-(a-5b+10c-10d+5e-f)$.

For the nth order, we see that we must have the following formula, in which D_n stands for the nth order of differences, $D_n = \pm$
$$\left(a-nb+\frac{n(n-1)}{2}c-\frac{n(n-1)(n-2)}{2\ .\ 3}d+\frac{n(n-1)(n-2)(n-3)}{2\ .\ 3\ .\ 4}e-\ldots\right)$$
The upper sign before the parenthesis must be used when n is even, and the lower one when n is odd.

PROBLEM

(450.) 2. To find the nth term of a series.

SOLUTION.

By Prob. 1. **(449)** we have the following equations,
$$D_1=b-a,$$
$$D_2=c-2b+a,$$
$$D_3=d-3c+3b-a,$$
$$D_4=e-4d+6c-4b+a,$$
$$D_5=f-5e+10d-10c+5b-a,$$
&c., &c.

Whence may be obtained
$$b=a+D_1,$$
$$c=a+2D_1+D_2,$$
$$d=a+3D_1+3D_2+D_3,$$
$$e=a+4D_1+6D_2+4D_3+D_4,$$
$$f=a+5D_1+10D_2+10D_3+5D_4+D_5,$$
&c., &c.

We see that in the $(n+1)$th term of the series a, b, c, d, e, &c., the coefficients of a, D_1, D_2, D_3, D_4, &c., are the same as the coefficients in the expansion of $(x+y)^n$. Therefore, the $(n+1)$th term of the

series a, b, c, d, &c., is $a+nD_1+\frac{n(n-1)}{2}D_2+\frac{n(n-1)(n-2)}{2\ .\ 3}D_3+\frac{n(n-1)(n-2)(n-3)}{2\ .\ 3\ .\ 4}D_4+$, &c., from which we see that the nth term must be $a+(n-1)D_1+\frac{(n-1)(n-2)}{2}D_2+\frac{(n-1)(n-2)(n-3)}{2\ .\ 3}D_3+\frac{(n-1)(n-2)(n-3)(n-4)}{2\ .\ 3\ .\ 4}D_4$, &c., which is obtained by writing $n-1$ for n.

PROBLEM

(451.) 3. To find the sum of n terms of a series.

SOLUTION.

Let a, b, c, d, &c., be the proposed series, of which we seek to find the sum of n of its terms.

Taking the series 0, a, $a+b$, $a+b+c$, $a+b+c+d$, &c., we see that its $(n+1)$th term is equal to the sum of n terms of the series a, b, c, d, &c.

Representing the first term of each order of differences in the series 0, a, $a+b$, $a+b+c$, &c., respectively by D_1, D_2, D_3, D_4, &c., we have by the last problem for the $(n+1)$th term of the series 0, a, $a+b$, $a+b+c$, $a+b+c+d$, &c., or the sum of n terms of the series a, b, c, d, &c., $0+nD_1+\frac{n(n-1)}{2}D_2+\frac{n(n-1)(n-2)}{2\ .\ 3}D_3+\frac{n(n-1)(n-2)(n-3)}{2\ .\ 3\ .\ 4}D_4+$, &c.

If we wish to refer the first term of each order of differences to the proposed series a, b, c, d, &c., instead of to the assumed series 0, a, $a+b$, $a+b+c$, $a+b+c+d$, &c., the formula becomes (since D_1, D_2, D_3, &c., of the assumed series are respectively equal to a, D_1, D_2, &c. of the proposed series) $S=na+\frac{n(n-1)}{2}D_1+\frac{n(n-1)(n-2)}{2\ .\ 3}D_2+\frac{n(n-1)(n-2)(n-3)}{2\ .\ 3\ .\ 4}D_3$, &c.

EXAMPLES.

1. What is the first term of the fourth order of differences of the series 1, 8, 27, 64, 125, &c.? *Ans.* 0.

2. What is the first term of the eighth order of differences of the series 1, 3, 9, 27, 81, &c.? *Ans.* 256.

3. What is the first term of the fifth order of differences of the series $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}$, &c.? *Ans.* $-\frac{1}{32}$.

4. What is the first term of the sixth order of differences of the series, 3, 6, 11, 17, 24, 36, 50, 72, &c.? *Ans.* -14.

5. What is the first term of the third order of differences of the series $1, 2^4, 3^4, 4^4$, &c.? *Ans.* 60.

6. What is the 10th term of the series 1, 4, 8, 13, 19, &c.? *Ans.* 64.

7. What is the 15th term of the series $1^2, 2^2, 3^2, 4^2$, &c.? *Ans.* 225.

8. What is the 20th term of the series 1, 3, 5, 7, &c.? *Ans.* 39.

9. What is the nth term of the series $a, a+d, a+2d, a+3d$, &c.? *Ans.* $a+(n-1)d$.

10. What is the nth term of the series 1, 3, 6, 10, 15, 21, &c.? *Ans.* $\frac{n(n+1)}{2}$.

11. What is the 50th term of the series 1, 4, 8, 13, &c.? *Ans.* 1324.

12. What is the sum of n terms of the series 1, 3, 5, 7, 9, &c.? *Ans.* n^2.

13. What is the sum of n terms of the series $1^2, 2^2, 3^2, 4^2$, &c.? *Ans.* $\frac{n(n+1)(2n+1)}{1\,.\,2\,.\,3}$.

14. What is the sum of n terms of the series 1, 2, 3, 4, 5, &c.? *Ans.* $\frac{n(n+1)}{2}$.

15. What is the sum of n terms of the series $1^3, 2^3, 3^3, 4^3$, &c.? *Ans.* $\frac{1}{4}(n^4+2n^3+n^2)$.

16. What is the sum of n terms of the series $1, 2^4, 3^4, 4^4$, &c.? *Ans.* $\frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}$.

17. What is the sum of n terms of the series 1, 8, 27, 64, &c.? *Ans.* $\frac{n^2(n+1)^2}{4}$.

18. What is the sum of n terms of the series 1, 3, 6, 10, 15, &c.?

Ans. $\dfrac{n(n+1)(n+2)}{1 \, . \, 2 \, . \, 3}$.

19. What is the sum of n terms of the series $1(m+1)$, $2(m+2)$, $3(m+3)$, $4(m+4)$, &c.?

Ans. $\dfrac{n(n+1)(1+2n+3m)}{1 \, . \, 2 \, . \, 3}$.

APPLICATION OF THE DIFFERENTIAL METHOD.

PROBLEM.

(452.) To find the number of balls or shells in a triangular pile.

SOLUTION.

By an inspection of a triangular pile of balls, or shells, we see that the first tier, commencing at the top, has one ball; the second, $1+2$; the third, $1+2+3$; the 4th, $1+2+3+4$; and the nth, $1+2+3+4+5 \ldots +n$. The number of tiers equals the number of balls, or shells, in one side of the base.

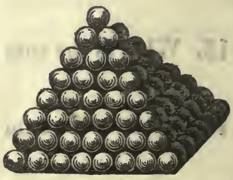
Then, to find the number of balls, or shells, in a triangular pile of n tiers, we must find the sum of the series 1, 3, 6, 10, ... to n terms, which is

$$S=\frac{n(n+1)(n+2)}{1 \, . \, 2 \, . \, 3}$$

PROBLEM.

(453.) To find the number of balls or shells in a square pile.

SOLUTION.



By an inspection of a square pile of balls, or shells, we see that the first tier, commencing at the top, has 1^2; the second, 2^2; the third, 3^2; the fourth, 4^2; and the nth, n^2 balls, or shells.

Then, to find the number of balls, or shells, in a square pile of n tiers, we must find the sum of the series 1^2, 2^2, 3^2, 4^2, 5^2, to n terms, which is

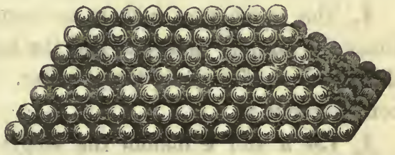
$$S=\frac{n(n+1)(2n+1)}{1 \, . \, 2 \, . \, 3}$$

PROBLEM.

(454.) To find the number of balls or shells, in an oblong pile.

SOLUTION.

By an inspection of an oblong pile of balls, or shells, we see that if $m+1$ represents the number in the first tier, commencing at the top, the number in the second is $2(m+2)$; in the 3d, $3(m+3)$; in the 4th, $4(m+4)$; and in the nth, $n(m+n)$.



Then, to find the number of balls, or shells, in an oblong pile of n tiers, we must find the sum of $1(m+1)$, $2(m+2)$, $3(m+3)$, ... to n terms, which is

$$S=\frac{n(n+1)(1+2n+3m)}{1\ .\ 2\ .\ 3}$$

(455.) These three formulas may be written as follows:

In the triangular pile, $S=\frac{1}{3}\cdot\frac{n(n+1)}{2}\cdot(n+1+1)$

" " square " $S=\frac{1}{3}\cdot\frac{n(n+1)}{2}\cdot(n+n+1)$

" " oblong " $S=\frac{1}{3}\cdot\frac{n(n+1)}{2}\cdot\{(n+m)+(n+m)+(m+1)\}$

Since $\frac{n(n+1)}{2}$ represents the number of balls, or shells, in the triangular face of each pile, and the other factor the number in the longest base line, plus the number in the line parallel to it, plus the number in the top tier, we have the following

GENERAL RULE.

Multiply one-third the number in the triangular face by the number in the longest base line, increased by the number in the opposite parallel line and by the number in the top tier, and the product will be the whole number of balls in the pile.

EXAMPLES.

1. How many balls in a triangular pile of 15 courses?

Ans. 680.

2. A complete square pile of shells has 14 courses: how many shells are in the pile, and how many remain after the removal of 5 courses? *Ans.* 1015 in the pile, and 960 remain.

3. In an incomplete oblong pile of balls, the length and breadth at the bottom are respectively 46 and 20, and the length and breadth at the top are 35 and 9. How many does it contain? *Ans.* 7190.

4. The number of balls in an incomplete square pile is equal to 6 times the number removed, and the number of courses left is equal to the number of courses taken away. How many balls were in the complete pile? *Ans.* 385.

5. Let h and k denote the length and breadth at the top of an oblong truncated pile, and n the number of balls in each of the slanting edges. How many in this truncated pile?

$$Ans.\ \frac{n}{6}\left\{2n^2+3n(h+k)+6hk-3(h+k+n)+1\right\}.$$

6. How many shot in a triangular pile whose bottom row contains 8 shot? *Ans.* 120.

7. How many shot in a square pile whose bottom row contains 8 shot? *Ans.* 204.

8. How many shot in an oblong pile whose length and breadth at the bottom contains respectively 16 and 7? *Ans.* 392.

9. How many shot in a triangular pile whose bottom row contains 30 shot? *Ans.* 4960.

10. How many shot in a square pile whose bottom row contains 30 shot? *Ans.* 9455.

11. How many shot in an oblong pile whose number of courses is 30, and top row 31? *Ans.* 23405.

12. How many shot in an incomplete oblong pile whose length and breadth at the bottom are respectively 46 and 20, and at the top 35 and 9? *Ans.* 7190.

SPECIAL SERIES.

THEOREM.

(456.) *Any fraction of the form* $\frac{q}{n(n+p)}$ *is equal to* $\frac{1}{p}$ *of the difference between the fractions* $\frac{q}{n}$ *and* $\frac{q}{n+p}$.

DEMONSTRATION.

$$\frac{q}{n}-\frac{q}{n+p}=\frac{qn+pq-nq}{n(n+p)}=\frac{pq}{n(n+p)}; \therefore \frac{q}{n(n+p)}=\frac{1}{p}\left(\frac{q}{n}-\frac{q}{n+p}\right).$$

Q. E. D.

COROLLARY.—If the *difference* between $\frac{q}{n}$ and $\frac{q}{n+p}$ is known, the value of $\frac{q}{n(n+p)}$ will be known, whether $\frac{q}{n}$ and $\frac{q}{n+p}$ are known or not.

Hence, we obtain the following

PRINCIPLE.

In any series of fractions having the form $\frac{q}{n(n+p)}$ *the sum of the series is equal to* $\frac{1}{p}$th *of the difference between a series of fractions of the form* $\frac{q}{n}$, *and a series of fractions of the form* $\frac{q}{n+p}$.

QUESTION.

(457.) What is the sum of the series $\frac{1}{1{\cdot}2}+\frac{1}{2{\cdot}3}+\frac{1}{3{\cdot}4}+\ldots$ to infinity?

SOLUTION.

Here $\frac{q}{n(n+p)}=\frac{1}{1{\cdot}2}$; $\therefore$ $q=1$; $n=1$; $n+p=2$, or $p=1$.

$\frac{q}{n(n+p)}=\frac{1}{2{\cdot}3}$; $\therefore$ $q=1$; $n=2$; $n+p=3$, or $p=1$.

$\frac{q}{n(n+p)}=\frac{1}{3{\cdot}4}$; $\therefore$ $q=1$; $n=3$; $n+p=4$, or $p=1$.

From which, we see that q constantly equals 1 and p constantly equals 1, and n successively equals 1, 2, 3, &c.

Therefore, $\frac{q}{n}=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\ldots$ to infinity.

and $\frac{q}{n+p}=\frac{1}{1+1}+\frac{1}{2+1}+\frac{1}{3+1}\ldots$ to infinity.

$$\frac{q}{n}-\frac{q}{n+p}=1.$$

Whence, $\frac{1}{p}\left(\frac{q}{n}-\frac{q}{n+p}\right)$, or $\frac{1}{p}$ of 1, which $=1$ is the sum of a series of fractions of the form $\frac{q}{n(n+p)}$, or of the series $\frac{1}{1{\cdot}2}+\frac{1}{2{\cdot}3}+\frac{1}{3{\cdot}4}+\ldots$ to infinity.

NOTE.—In the following examples some difficulties are purposely left for the student to overcome. But it is believed that the skill that he has acquired by mastering the work thus far will enable him to solve them.

QUESTIONS.

1. What is the sum of the series $\frac{1}{1\cdot3}+\frac{1}{2\cdot4}+\frac{1}{3\cdot5}+\ldots\ldots$ to infinity?

Ans. $\frac{3}{4}$.

2. What is the sum of the series $\frac{1}{1\cdot3}-\frac{1}{2\cdot4}+\frac{1}{3\cdot5}-\ldots$ to infinity?

Ans. $\frac{1}{4}$.

3. What is the sum of the series $\frac{4}{1\cdot5}+\frac{4}{5\cdot9}+\frac{4}{9\cdot13}+\frac{4}{13\cdot17}+\ldots$ to infinity?

Ans. 1.

4. What is the sum of the series $\frac{1}{1\cdot4}+\frac{1}{2\cdot5}+\frac{1}{3\cdot6}+\frac{1}{4\cdot7}+\ldots\ldots$ to infinity?

Ans. $\frac{11}{18}$.

5. What is the sum of the series $\frac{1}{1\cdot3}+\frac{1}{3\cdot5}+\frac{1}{5\cdot7}+\ldots\ldots$ to infinity?

Ans. $\frac{1}{2}$.

6. What is the sum of n terms of the series $\frac{1}{1\cdot4}+\frac{1}{2\cdot5}+\frac{1}{3\cdot6}+\frac{1}{4\cdot7}+$ &c?

Ans. $\frac{n}{3n+3}+\frac{n}{6n+12}+\frac{n}{9n+27}$.

7. What is the sum of n terms of the series $\frac{1}{1\cdot3}+\frac{1}{3\cdot5}+\frac{1}{5\cdot7}+$, &c.

Ans. $\frac{n}{2n+1}$.

8. What is the sum of the series $\frac{2}{3\cdot5}-\frac{3}{5\cdot7}+\frac{4}{7\cdot9}-\frac{5}{9\cdot11}+\ldots$ to infinity?

Ans. $\frac{1}{12}$.

9. What is the sum of the series $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+$, &c., to infinity?

Ans. 2.

10. What is the sum of the series $\frac{1}{3\cdot8}+\frac{1}{6\cdot12}+\frac{1}{9\cdot16}+$, &c., to infinity?

Ans. $\frac{1}{12}$.

11. What is the sum of n terms of the series $\frac{1}{3\cdot 8}+\frac{1}{6\cdot 12}+\frac{1}{9\cdot 16}+$, &c.?

Ans. $\frac{1}{12}\left(\frac{n}{n+1}\right)$.

12. What is the sum of n terms of the series $\frac{2}{3\cdot 5}-\frac{3}{5\cdot 7}+\frac{4}{7\cdot 9}-\frac{5}{9\cdot 11}+$, &c.? *Ans.* $\frac{1}{12}\pm\frac{1}{4(2n+3)}$, according as n is odd or even.

By reasoning as in **(456)**, we obtain the following

PRINCIPLE.

(458.) *If any series of fractions have the form* $\frac{q}{n(n+p)(n+2p)}$, *the sum of the series is equal to* $\frac{1}{2p}$ *of the difference between a series of fractions of the form* $\frac{q}{n(n+p)}$ *and a series of fractions of the form* $\frac{q}{(n+p)(n+2p)}$.

QUESTIONS.

1. What is the sum of the series $\frac{1}{1\cdot 3\cdot 5}+\frac{2}{3\cdot 5\cdot 7}+\frac{3}{5\cdot 7\cdot 9}+$, &c., to infinity. *Ans.* $\frac{1}{8}$.

2. What is the sum of the series $\frac{1}{1\cdot 3\cdot 5}+\frac{4}{3\cdot 5\cdot 7}+\frac{7}{5\cdot 7\cdot 9}+\frac{10}{7\cdot 9\cdot 11}+$, &c., to infinity? *Ans.* $\frac{5}{24}$.

3. What is the sum of the series $\frac{3}{5\cdot 8\cdot 11}+\frac{9}{8\cdot 11\cdot 14}+\frac{15}{11\cdot 14\cdot 17}+$, &c., to infinity? *Ans.* $\frac{13}{240}$.

4. What is the sum of the series $\frac{4}{1\cdot 2\cdot 3}+\frac{5}{2\cdot 3\cdot 4}+\frac{6}{3\cdot 4\cdot 5}+$, &c., to infinity? *Ans.* $1\frac{1}{4}$.

5. What is the sum of the series

$$\frac{a}{n(n+p)(n+2p)}+\frac{a+b}{(n+p)(n+2p)(n+3p)}+\frac{a+2b}{(n+2p)(n+3p)(n+4p)},$$

&c., to infinity? *Ans.* $\frac{pa+bn}{2p^2n(n+p)}$.

Again, by reasoning as in (**456**), we obtain the following

PRINCIPLE.

(459.) *If any series of fractions have the form*

$$\frac{q}{n(n+p)(n+2p)n+3p)},$$

the sum of the series is equal to $\frac{1}{3p}$ *of the difference between a series of fractions of the form* $\frac{q}{n(n+p)(n+2p)}$, *and a series of fractions of the form* $\frac{q}{n(n+p)(n+2p)(n+3p)}$.

QUESTIONS.

1. What is the sum of the series $\frac{1}{1\cdot2\cdot3\cdot4}+\frac{1}{2\cdot3\cdot4\cdot5}+\frac{1}{3\cdot4\cdot5\cdot6}+$, &c., to infinity? *Ans.* $\frac{1}{18}$.

2. What is the sum of the series $\frac{1}{1\cdot3\cdot5\cdot7}+\frac{2}{3\cdot5\cdot7\cdot9}+\frac{3}{5\cdot7\cdot9\cdot11}+$, &c., to infinity? *Ans.* $\frac{1}{72}$.

3. What is the sum of the series $\frac{2}{3\cdot6\cdot9\cdot12}+\frac{5}{6\cdot9\cdot12\cdot15}+\frac{8}{9\cdot12\cdot15\cdot18}+$, &c., to infinity? *Ans.* $\frac{21}{8745}$.

4. What is the sum of the series $\frac{6^2}{1\cdot2\cdot3\cdot4}+\frac{7^2}{2\cdot3\cdot4\cdot5}+\frac{8^2}{3\cdot4\cdot5\cdot6}+$, &c., to infinity? *Ans.* $\frac{89}{36}$.

Again reasoning as in (**456**), we obtain the following

PRINCIPLE.

(460.) *In any series of fractions of the form*

$$\frac{q}{n(n+p)(n+2p)\ .\ .\ .\ .\ (n+mp)},$$

the sum of the series is equal to the $\frac{1}{mp}$ *of the difference between a series of fractions of the form* $\frac{q}{n(n+p)(n+2p)\ .\ .\ .\ .\ \big(n+(m-1)p\big)}$, *and a series of fractions of the form* $\frac{q}{(n+p)(n+2p)\ .\ .\ .\ (n+mp)}$.

Again reasoning as in (**456**), we obtain the following

PRINCIPLE.

(461.) *In any series of fractions of the form*

$$\frac{a(a+b)\ \ldots\ (a+pb)}{n(n+b)\ \ldots\ (n+pb)},$$

the sum of the series is equal to $\frac{1}{n-a-b}$ *of the difference between a series of fractions of the form* $\frac{a(a+b)\ \ldots\ (a+pb)}{n(n+b)\ \ldots\ [n+(p-1)b]}$, *and a series of fractions of the form* $\frac{a(a+b)\ \ldots\ [(a+(p+1)b)]}{n(n+b)\ \ldots\ (n+pb)}$.

QUESTIONS.

1. What is the sum of r terms of the series $\frac{1}{2}+\frac{1\cdot3}{2\cdot4}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}+\frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}+$, &c.? *Ans.* $\frac{1\cdot3\cdot5\cdot7\ \ldots\ (2r+1)}{2\cdot4\cdot6\ \ldots\ 2r}-1$.

2. What is the sum of r terms of the series $\frac{2}{3}+\frac{2\cdot4}{2\cdot5}+\frac{2\cdot4\cdot6}{3\cdot5\cdot7}+\frac{2\cdot4\cdot6\cdot8}{3\cdot5\cdot7\cdot9}+$, &c.? *Ans.* $\frac{2\cdot4\cdot6\cdot8\ \ldots\ (2r+2)}{3\cdot5\cdot7\cdot9\ \ldots\ (2r+1)}-2$.

3. What is the sum of the series $\frac{2}{5\cdot6}+\frac{2\cdot3}{5\cdot6\cdot7}+\frac{2\cdot3\cdot4}{5\cdot6\cdot7\cdot8}+$, &c., to infinity? *Ans.* $\frac{2}{15}$.

4. What is the sum of r terms of the series $\frac{a}{n}+\frac{a(a+b)}{n(n+b)}+\frac{a(a+b)(a+2b)}{n(n+b)(n+2b)}+$, &c.?

$$Ans.\ \frac{1}{n-a-b}\left(a-\frac{a(a+b)(a+2b)\ \ldots\ (a+rb)}{n(n+b)(n+2b)\ \ldots\ [n+(r-1)b]}\right).$$

REMARK.—If $n=a+2b$, whence $n+b=a+3b$ and $n+2b=a+4b$, &c., and $n+(r-2)b=a+rb$, the fraction in the parenthesis becomes

$$\frac{a(a+b)}{a+(1+r)b},$$

which vanishes when r is infinite, and we therefore have for the sum of the series to infinity

$$\frac{a}{n-a-b}.$$

When $n=a+b$, the fraction in the parenthesis becomes a, and the sum of the series $\frac{1}{n-a-b}(a-a)=\frac{1}{0}(0)=\frac{0}{0}$, an expression of no definite signification in its present form.

It may be observed that this is obtained without reference to the value of r; hence, whether r is infinite or finite, the result is the same.

(462.) Some series may be very beautifully summed as follows.

QUESTION

1. What is the sum of the infinite series $x+x^2+x^3+x^4+$, &c.?

SOLUTION.

$$\begin{aligned} &\text{Let} \quad S=x+x^2+x^3+x^4+x^5+, \text{\&c.} \\ &\text{then} \quad \underline{xS= \quad +x^2+x^3+x^4+x^5+, \text{\&c.}} \\ &\therefore \quad (1-x)S=x. \\ &\text{whence} \quad S=\frac{x}{1-x}. \end{aligned}$$

QUESTION

2. What is the sum of the infinite series $x-x^2+x^3-x^4+$, &c.?

SOLUTION.

$$\begin{aligned} &\text{Let} \quad S=x-x^2+x^3-x^4+, \text{\&c.} \\ &\text{then} \quad \underline{-xS= \quad -x^2+x^3-x^4+, \text{\&c.}} \\ &\therefore \quad (1+x)S=x. \\ &\text{whence,} \quad S=\frac{x}{1+x}. \end{aligned}$$

QUESTION

3. What is the sum of the infinite series $x+2x^2+3x^3+4x^4+$, &c.?

SOLUTION.

$$\begin{aligned} &\text{Let} \quad S=x+2x^2+3x^3+4x^4+, \text{\&c.} \\ &\text{then} \quad -2xS= \quad -2x^2-4x^3-6x^4-, \text{\&c.} \\ &\text{and} \quad x^2S= \quad \quad + \ x^3+2x^4+, \text{\&c.} \\ &\therefore \quad (1-2x+x^2)=x. \\ &\text{whence,} \quad S=\frac{x}{1-2x+x^2}. \end{aligned}$$

QUESTION

4. What is the sum of the infinite series $x+4x^2+9x^3+16x^4+$, &c.?

SOLUTION.

$$\begin{aligned} \text{Let}\quad S&=x+4x^2+9x^3+16x^4+,\ \&c. \\ \text{then}\quad -3xS&=-3x^2-12x^3-27x^4-,\ \&c. \\ \text{and}\quad 3x^2S&=+3x^3+12x^4+,\ \&c. \\ \text{and}\quad -x^3S&=-x^4-,\ \&c. \end{aligned}$$

$$\therefore\ (1-3x+3x^2-x^3)S=x+x^2.$$

$$\text{whence,}\quad S=\frac{x(1+x)}{(1-x)^3}.$$

PROMISCUOUS QUESTIONS.

1. What is the sum of the infinite series $1+x+x+x^3+$ &c.?

Ans. $\frac{1}{1-x}$.

2. What is the sum of the infinite series $1+2x+3x^2+4x^3+5x^4+$, &c.?

Ans. $\frac{1}{(1-x)^2}$.

3. What is the sum of the infinite series $\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\frac{1}{3\cdot4\cdot5}+$, &c.?

Ans. $\frac{1}{4}$.

4. What is the sum of the infinite series $\frac{3}{1\cdot2\cdot2}+\frac{4}{2\cdot3\cdot2^2}+\frac{5}{3\cdot4\cdot2^3}+$, &c.?

Ans. 1.

5. What is the sum of the infinite series $\frac{5}{1\cdot2\cdot3\cdot2^2}+\frac{6}{2\cdot3\cdot4\cdot2^3}+\frac{7}{3\cdot4\cdot5\cdot2^4}+$, &c.?

Ans. $\frac{1}{4}$.

6. What is the sum of the infinite series $\frac{1}{8\cdot18}+\frac{1}{10\cdot21}+\frac{1}{12\cdot24}+\frac{1}{14\cdot27}+$, &c.?

Ans. $\frac{3}{80}$.

7. What is the sum of the infinite series $\frac{10\cdot18}{2\cdot4\cdot9\cdot12}+\frac{12\cdot21}{4\cdot6\cdot12\cdot15}+\frac{14\cdot24}{6\cdot8\cdot15\cdot18}+$, &c.?

Ans. $\frac{19}{54}$.

8. What is the sum of the infinite series $x+3x^2+6x^3+10x^4+$, &c.?

Ans. $\frac{x}{(1-x)^3}$.

9. What is the sum of n terms of the series $\frac{1}{4 \cdot 8}-\frac{1}{6 \cdot 10}+\frac{1}{8 \cdot 12}-$, &c.?

Ans. $\frac{n}{16(1+n)}-\frac{n}{12(3+2n)}$.

10. What is the sum of the infinite series $\frac{4}{1 \cdot 5}+\frac{4}{5 \cdot 9}+\frac{4}{9 \cdot 13}+\frac{4}{13 \cdot 17}+\frac{4}{17 \cdot 21}+$, &c.

Ans. 1.

THE END.